Contents 19

Differential Equations

Learning outcomes

In this Workbook you will learn what a differential equation is and how to recognise some of the basic different types. You will learn how to apply some common techniques used to obtain general solutions of differential equations and how to fit initial or boundary conditions to obtain a unique solution. You will appreciate how differential equations arise in applications and you will gain some experience in applying your knowledge to model a number of engineering problems using differential equations.

Modelling with Differential Equations 19.1

Introduction

Many models of engineering systems involve the **rate of change** of a quantity. There is thus a need to incorporate derivatives into the mathematical model. These mathematical models are examples of **differential equations**.

Accompanying the differential equation will be one or more conditions that let us obtain a unique solution to a particular problem. Often we solve the differential equation first to obtain a general solution; then we apply the conditions to obtain the unique solution. It is important to know which conditions must be specified in order to obtain a unique solution.

Prerequisites

Before starting this Section you should ...

- be able to differentiate; (HELM 11)
- be able to integrate; (HELM 13)

Learning Outcomes

On completion you should be able to ...

- understand the use of differential equations in modelling engineering systems
- identify the order and type of a differential equation
- recognise the nature of a general solution
- determine the nature of the appropriate additional conditions which will give a unique solution to the equation

1. Case study: Newton's law of cooling

When a hot liquid is placed in a cooler environment, experimental observation shows that its temperature decreases to approximately that of its surroundings. A typical graph of the temperature of the liquid plotted against time is shown in Figure 1.

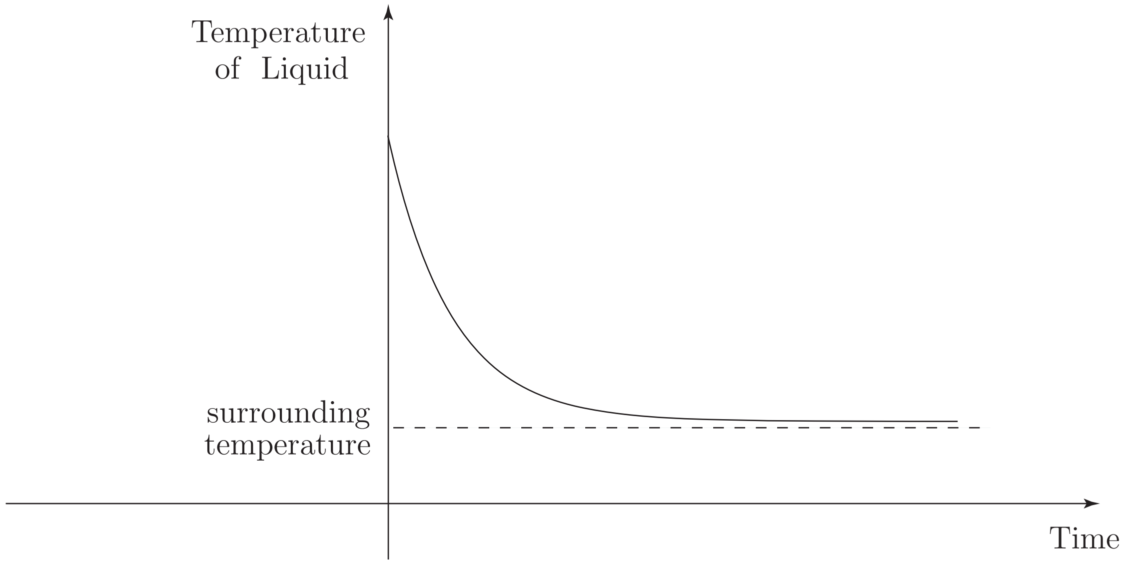


Figure 1

After an initially rapid decrease the temperature changes progressively less rapidly and eventually the curve appears to 'flatten out'.

Newton's law of cooling states that the rate of cooling of liquid is proportional to the difference between its temperature and the temperature of its environment (the ambient temperature). To convert this into mathematics, let t be the time elapsed (in seconds, s), θ the temperature of the liquid (°C),and θ_0 the temperature of the liquid at the start $(t = 0)$. The temperature of the surroundings is denoted by θ_{s}.

Write down the mathematical equation which is equivalent to Newton's law of cooling and state the accompanying condition.

First, find an expression for the rate of cooling, and an expression for the difference between the liquid's temperature and that of the environment:

Your solution

Answer

The rate of cooling is the rate of change of temperature with time: $\dfrac{d\theta}{dt}$.

The temperature difference is $\theta - \theta_{\mathsf{s}}$.

Now formulate Newton's law of cooling:

Your solution

Answer
You should obtain $\dfrac{d\theta}{dt} \propto (\theta - \theta_{\mathsf{s}})$ or, equivalently: $\dfrac{d\theta}{dt} = -k(\theta - \theta_{\mathsf{s}})$. k is a positive constant of proportion and the negative sign is present because $(\theta - \theta_{\mathsf{s}})$ is positive, whereas $\dfrac{d\theta}{dt}$ must be negative, since θ decreases with time. The units of k are s^{-1}. The accompanying condition is $\theta = \theta_0$ at $t = 0$ which simply states the temperature of the liquid when the cooling begins.

In the above Task we call t the independent variable and θ the dependent variable. Since the condition is given at $t = 0$ we refer to it as an initial condition. For future reference, the solution of the above differential equation which satisfies the initial condition is $\theta = \theta_{\mathsf{s}} + (\theta_0 - \theta_{\mathsf{s}})\mathrm{e}^{-kt}$.

2. The general solution of a differential equation

Consider the equation $y = A\mathrm{e}^{2x}$ where A is an arbitrary constant. If we differentiate it we obtain

$$\frac{dy}{dx} = 2A\mathrm{e}^{2x}$$

and so, since $y = A\mathrm{e}^{2x}$ we obtain

$$\frac{dy}{dx} = 2y.$$

Thus a differential equation satisfied by y is

$$\frac{dy}{dx} = 2y.$$

Note that we have eliminated the arbitrary constant.
Now consider the equation

$$y = A\cos 3x + B\sin 3x$$

where A and B are arbitrary constants. Differentiating, we obtain

$$\frac{dy}{dx} = -3A\sin 3x + 3B\cos 3x.$$

Differentiating a second time gives

$$\frac{d^2y}{dx^2} = -9\,A\cos 3x - 9\,B\sin 3x.$$

The right-hand side is simply (-9) times the expression for y. Hence y satisfies the differential equation

$$\frac{d^2y}{dx^2} = -9y.$$

Task

Find a differential equation satisfied by $y = A\cosh 2x + B\sinh 2x$ where A and B are arbitrary constants.

Your solution

Answer

Differentiating once we obtain $\dfrac{dy}{dx} = 2A\sinh 2x + 2B\cosh 2x$

Differentiating a second time we obtain $\dfrac{d^2y}{dx^2} = 4A\cosh 2x + 4B\sinh 2x$

Hence $\dfrac{d^2y}{dx^2} = 4y$

We have seen that an expression including one arbitrary constant required one differentiation to obtain a differential equation which eliminated the arbitrary constant. Where two constants were present, two differentiations were required. Is the converse true? For example, would a differential equation involving $\dfrac{dy}{dx}$ as the only derivative have a general solution with one arbitrary constant and would a differential equation which had $\dfrac{d^2y}{dx^2}$ as the highest derivative produce a general solution with two arbitrary constants? The answer is, usually, yes.

Task

Integrate twice the differential equation

$$\frac{d^2y}{dx^2} = \frac{w}{2}(\ell x - x^2),$$

where w and ℓ are constants, to find a general solution for y.

Your solution

Answer

Integrating once: $\dfrac{dy}{dx} = \dfrac{w}{2}\left(\dfrac{\ell x^2}{2} - \dfrac{x^3}{3}\right) + A$ where A is an arbitrary constant (of integration).

Integrating again: $y = \dfrac{w}{2}\left(\dfrac{\ell x^3}{6} - \dfrac{x^4}{12}\right) + A\,x + B$ where B is a second arbitrary constant.

Consider the simple differential equation

$$\frac{dy}{dx} = 2x.$$

On integrating, we obtain the general solution

$$y = x^2 + C$$

where C is an arbitrary constant. As C varies we get different solutions, each of which belongs to the family of solutions. Figure 2 shows some examples.

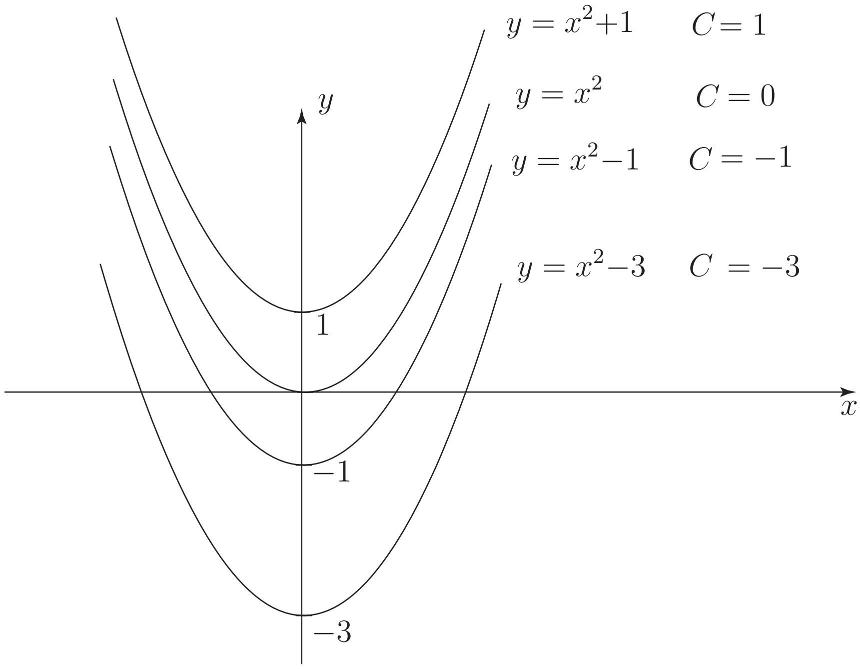


Figure 2

It can be shown that no two members of this family of graphs ever meet and that through each point in the x-y plane passes one, and only one, of these graphs. Hence if we specify the boundary condition $y = 2$ when $x = 0$, written $y(0) = 2$, then using $y = x^2 + c$:

$$2 = 0 + C \quad \text{so that} \quad C = 2$$

and $y = x^2 + 2$ is the unique solution.

Task

Find the unique solution of the differential equation $\dfrac{dy}{dx} = 3x^2$ which satisfies the condition $y(1) = 4$.

Your solution

Answer

You should obtain $y = x^3 + 3$ since, by a single integration we have $y = x^3 + C$, where C is an arbitrary constant. Now when $x = 1$, $y = 4$ so that $4 = 1 + C$. Hence $C = 3$ and the unique solution is $y = x^3 + 3$.

Example 1

Solve the differential equation $\dfrac{d^2y}{dx^2} = 6x$ subject to the conditions

(a) $y(0) = 2$ and $y(1) = 3$

(b) $y(0) = 2$ and $y(1) = 5$

(c) $y(0) = 2$ and $\dfrac{dy}{dx} = 1$ at $x = 0$.

Solution

(a) Integrating the differential equation once produces $\dfrac{dy}{dx} = 3x^2 + A$. The general solution is found by integrating a second time to give $y = x^3 + Ax + B$, where A and B are arbitrary constants.

Imposing the conditions $y(0) = 2$ and $y(1) = 3$: at $x = 0$ we have $y = 2 = 0 + 0 + B = B$ so that $B = 2$, and at $x = 1$ we have $y = 3 = 1 + A + B = 1 + A + 2$. Therefore $A = 0$ and the solution is

$$y = x^3 + 2.$$

(b) Here the second condition is $y(1) = 5$ so at $x = 1$

$$y = 5 = 1 + A + 2 \qquad \text{so that} \quad A = 2$$

and the solution in this case is

$$y = x^3 + 2x + 2.$$

(c) Here the second condition is

$$\frac{dy}{dx} = 1 \text{ at } x = 0 \qquad \text{i.e.} \qquad y'(0) = 1$$

then since $\dfrac{dy}{dx} = 3x^2 + A$, putting $x = 0$ we get:

$$\frac{dy}{dx} = 1 = 0 + A$$

so that $A = 1$ and the solution in this case is $y = x^3 + x + 2$.

3. Classifying differential equations

When solving differential equations (either analytically or numerically) it is important to be able to recognise the various kinds that can arise. We therefore need to introduce some terminology which will help us to distinguish one kind of differential equation from another.

- An **ordinary differential equation** (ODE) is any relation between a function of a **single variable** and its derivatives. (All differential equations studied in this workbook are ordinary.)

- The **order** of a differential equation is the order of the highest derivative in the equation.

- A differential equation is **linear** if the dependent variable and its derivatives occur to the first power only and if there are no products involving the dependent variable or its derivatives.

Example 2

Classify the differential equations specifying the order and type (linear/non-linear)

(a) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = x^2$

(b) $\dfrac{d^2x}{dt^2} = \left(\dfrac{dx}{dt}\right)^3 + 3x$

(c) $\dfrac{dx}{dt} - x = t^2$

(d) $\dfrac{dy}{dt} + \cos y = 0$

(e) $\dfrac{dy}{dt} + y^2 = 4$

Solution

(a) Second order, linear.

(b) Second order, non-linear (because of the cubic term).

(c) First order, linear.

(d) First order, non-linear (because of the $\cos y$ term).

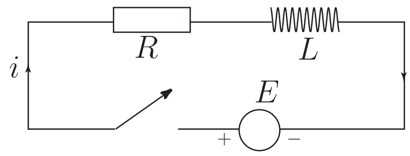
(e) First order, non-linear (because of the y^2 term).

Note that in (a) the independent variable is x whereas in the other cases it is t.

In (a), (d) and (e) the dependent variable is y and in (b) and (c) it is x.

Exercises

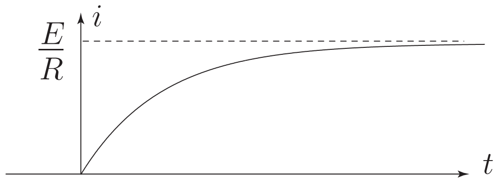
1. In this RL circuit the switch is closed at $t = 0$ and a constant voltage E is applied.



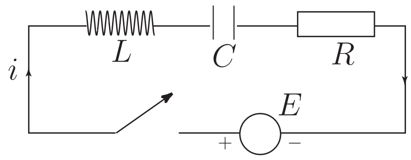
The voltage across the resistor is iR where i is the current flowing in the circuit and R is the (constant) resistance. The voltage across the inductance is $L\dfrac{di}{dt}$ where L is the constant inductance.

Kirchhoff's law of voltages states that the applied voltage is the sum of the other voltages in the circuit. Write down a differential equation for the current i and state the initial condition.

2. The diagram below shows the graph of i against t (from Exercise 1). What information does this graph convey?



3. In the LCR circuit below the voltage across the capacitor is q/C where q is the charge on the capacitor, and C is the capacitance. Note that $\dfrac{dq}{dt} = i$. Find a differential equation for i and write down the initial conditions if the initial charge is zero and the switch is closed at $t = 0$.



4. Find differential equations satisfied by

   (a) $y = A\cos 4x + B\sin 4x$

   (b) $x = A\,\mathrm{e}^{-2t}$

   (c) $y = A\sin x + B\sinh x + C\cos x + D\cosh x$ (harder)

5. Find the family of solutions of the differential equation $\dfrac{dy}{dx} = -2x$. Sketch the curves of some members of the family on the same axes. What is the solution if $y(1) = 3$?

6. (a) Find the general solution of the differential equation $y'' = 12x^2$.

   (b) Find the solution which satisfies $y(0) = 2,\ y(1) = 8$

   (c) Find the solution which satisfies $y(0) = 1,\ y'(0) = -2$.

7. Classify the differential equations

   (a) $\dfrac{d^2x}{dt^2}+3\,\dfrac{dx}{dt} = x$ (b) $\dfrac{d^3y}{dx^3} = \left(\dfrac{dy}{dx}\right)^2+\dfrac{dy}{dx}$ (c) $\dfrac{dy}{dx}+y = \sin x$ (d) $\dfrac{d^2y}{dx^2}+y\dfrac{dy}{dx} = 2.$

Answers

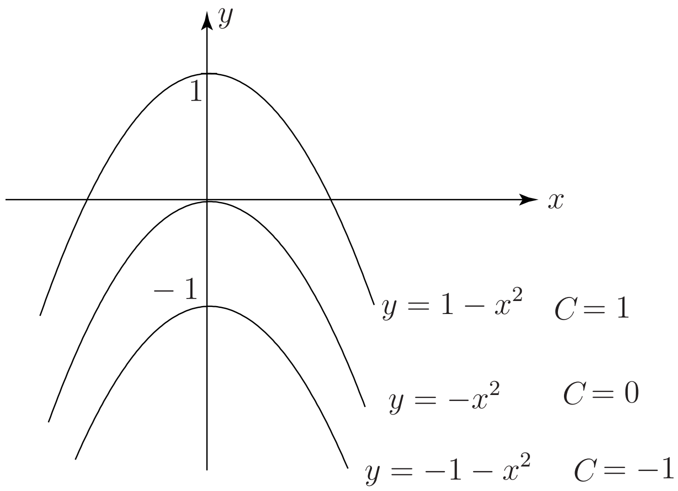
1. $L\,\dfrac{di}{dt} + R\,i = E$; $i = 0$ at $t = 0$.

2. Current increases rapidly at first, then less rapidly and tends to the value $\dfrac{E}{R}$ which is what it would be in the absence of L.

3. $L\,\dfrac{d^2q}{dt^2} + R\,\dfrac{dq}{dt} + \dfrac{q}{C} = E;$ $q = 0$ and $i = \dfrac{dq}{dt} = 0$ at $t = 0$.

4. (a) $\dfrac{d^2y}{dx^2} = -16y$ (b) $\dfrac{dx}{dt} = -2x$ (c) $\dfrac{d^4y}{dx^4} = y$

5. $y = -x^2 + C$



If $3 = -1 + C$ then $C = 4$ and $y = -x^2 + 4$.

6.

(a) $y = x^4 + A\,x + B$

(b) When $x = 0,\ y = 2 = B$; hence $B = 2$. When $x = 1,\ y = 8 = 1 + A + B = 3 + A$ hence $A = 5$ and $y = x^4 + 5x + 2$.

(c) When $x = 0$ $y = 1 = B$. Hence $B = 1$; $\dfrac{dy}{dx} = y' = 4x^3 + A$, so at $x = 0$, $y' = -2 = A$.

Therefore $y = x^4 - 2x + 1$

7. (a) Second order, linear

(b) Third order, non-linear *(squared term)*

(c) First order, linear

(d) Second order, non-linear *(product term)*

First Order Differential Equations 19.2

Introduction

Separation of variables is a technique commonly used to solve first order ordinary differential equations. It is so-called because we rearrange the equation to be solved such that all terms involving the dependent variable appear on one side of the equation, and all terms involving the independent variable appear on the other. Integration completes the solution. Not all first order equations can be rearranged in this way so this technique is not always appropriate. Further, it is not always possible to perform the integration even if the variables are separable.

In this Section you will learn how to decide whether the method is appropriate, and how to apply it in such cases.

An **exact** first order differential equation is one which can be solved by simply integrating both sides. Only very few first order differential equations are exact. You will learn how to recognise these and solve them. Some others may be converted simply to exact equations and that is also considered

Whilst exact differential equations are few and far between an important class of differential equations can be converted into exact equations by multiplying through by a function known as the **integrating factor** for the equation. In the last part of this Section you will learn how to decide whether an equation is capable of being transformed into an exact equation, how to determine the integrating factor, and how to obtain the solution of the original equation.

Prerequisites

Before starting this Section you should ...

- understand what is meant by a differential equation; (Section 19.1)

Learning Outcomes

On completion you should be able to ...

- explain what is meant by separating the variables of a first order differential equation
- determine whether a first order differential equation is separable
- solve a variety of equations using the separation of variables technique

1. Separating the variables in first order ODEs

In this Section we consider differential equations which can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

Note that the right-hand side is a product of a function of x, and a function of y. Examples of such equations are

$$\frac{dy}{dx} = x^2\,y^3, \qquad \frac{dy}{dx} = y^2\sin x \quad \text{and} \quad \frac{dy}{dx} = y\ln x$$

Not all first order equations can be written in this form. For example, it is not possible to rewrite the equation

$$\frac{dy}{dx} = x^2 + y^3$$

in the form

$$\frac{dy}{dx} = f(x)g(y)$$

Task

Determine which of the following differential equations can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

If possible, rewrite each equation in this form.

(a) $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$, (b) $\dfrac{dy}{dx} = 4x^2 + 2y^2$, (c) $y\dfrac{dy}{dx} + 3x = 7$

Your solution

Answer

(a) $\dfrac{dy}{dx} = x^2\left(\dfrac{1}{y^2}\right)$, (b) cannot be written in the stated form,

(c) Reformulating gives $\dfrac{dy}{dx} = (7-3x)\times\dfrac{1}{y}$ which is in the required form.

The variables involved in differential equations need not be x and y. Any symbols for variables may be used. Other first order differential equations are

$$\frac{dz}{dt} = t\mathrm{e}^{z} \qquad \frac{d\theta}{dt} = -\theta \quad \text{and} \quad \frac{dv}{dr} = v\left(\frac{1}{r^2}\right)$$

Given a differential equation in the form

$$\frac{dy}{dx} = f(x)g(y)$$

we can divide through by $g(y)$ to obtain

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

If we now integrate both sides of this equation with respect to x we obtain

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$$

that is

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

We have **separated the variables** because the left-hand side contains only the variable y, and the right-hand side contains only the variable x. We can now try to integrate each side separately. If we can actually perform the required integrations we will obtain a relationship between y and x. Examples of this process are given in the next subsection.

Key Point 1

Method of Separation of Variables

The solution of the equation

$$\frac{dy}{dx} = f(x)g(y)$$

may be found from separating the variables and integrating:

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

2. Applying the method of separation of variables to ODEs

Example 3

Use the method of separation of variables to solve the differential equation

$$\frac{dy}{dx} = \frac{3x^2}{y}$$

Solution

The equation already has the form

$$\frac{dy}{dx} = f(x)g(y)$$

where

$$f(x) = 3x^2 \qquad \text{and} \qquad g(y) = 1/y.$$

Dividing both sides by $g(y)$ we find

$$y\frac{dy}{dx} = 3x^2$$

Integrating both sides with respect to x gives

$$\int y\frac{dy}{dx}\,dx = \int 3x^2\,dx$$

that is

$$\int y\,dy = \int 3x^2\,dx$$

Note that the left-hand side is an integral involving just y; the right-hand side is an integral involving just x. After integrating both sides with respect to the stated variables we find

$$\tfrac{1}{2}y^2 = x^3 + c$$

where c is a constant of integration. (You might think that there would be a constant on the left-hand side too. You are quite right but the two constants can be combined into a single constant and so we need only write one.)

We now have a relationship between y and x as required. Often it is sufficient to leave your answer in this form but you may also be required to obtain an explicit relation for y in terms of x. In this particular case

$$y^2 = 2x^3 + 2c$$

so that

$$y = \pm\sqrt{2x^3 + 2c}$$

Task

Use the method of separation of variables to solve the differential equation

$$\frac{dy}{dx} = \frac{\cos x}{\sin 2y}$$

First separate the variables so that terms involving y and $\dfrac{dy}{dx}$ appear on the left, and terms involving x appear on the right:

Your solution

Answer
You should have obtained

$$\sin 2y \, \frac{dy}{dx} = \cos x$$

Now reformulate both sides as integrals:

Your solution

Answer

$$\int \sin 2y \, \frac{dy}{dx} \, dx = \int \cos x \, dx \quad \text{that is} \quad \int \sin 2y \, dy = \int \cos x \, dx$$

Now integrate both sides:

Your solution

Answer

$$-\tfrac{1}{2} \cos 2y = \sin x + c$$

Finally, rearrange to obtain an expression for y in terms of x:

Your solution

Answer

$$y = \tfrac{1}{2} \cos^{-1}(D - 2\sin x) \text{ where } D = -2c$$

Exercises

1. Solve the equation

$$\frac{dy}{dx} = \frac{\mathrm{e}^{-x}}{y}.$$

2. Solve the following equation subject to the condition $y(0) = 1$:

$$\frac{dy}{dx} = 3x^2\mathrm{e}^{-y}$$

3. Find the general solution of the following equations:

(a) $\dfrac{dy}{dx} = 3$, (b) $\dfrac{dy}{dx} = \dfrac{6\sin x}{y}$

4. (a) Find the general solution of the equation

$$\frac{dx}{dt} = t(x-2).$$

(b) Find the particular solution which satisfies the condition $x(0) = 5$.

5. Some equations which do not appear to be separable can be made so by means of a suitable substitution. By means of the substitution $z = y/x$ solve the equation

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x} + 1$$

6. The equation

$$iR + L\frac{di}{dt} = E$$

where R, L and E are constants arises in electrical circuit theory. This equation can be solved by separation of variables. Find the solution which satisfies the condition $i(0) = 0$.

Answers

1. $y = \pm\sqrt{D - 2\mathrm{e}^{-x}}$.

2. $y = \ln(x^3 + e)$.

3 (a) $y = 3x + C$, (b) $\frac{1}{2}y^2 = C - 6\cos x$.

4. (a) $x = 2 + A\mathrm{e}^{t^2/2}$, (b) $x = 2 + 3\mathrm{e}^{t^2/2}$.

5. $z = \tan(\ln Dx)$ so that $y = x\tan(\ln Dx)$.

6. $i = \dfrac{E}{R}(1 - \mathrm{e}^{-t/\tau})$ where $\tau = L/R$.

3. Exact equations

Consider the differential equation

$$\frac{dy}{dx} = 3x^2$$

By direct integration we find that the general solution of this equation is

$$y = x^3 + C$$

where C is, as usual, an arbitrary constant of integration.
Next, consider the differential equation

$$\frac{d}{dx}(yx) = 3x^2.$$

Again, by direct integration we find that the general solution is

$$yx = x^3 + C.$$

We now divide this equation by x to obtain

$$y = x^2 + \frac{C}{x}.$$

The differential equation $\frac{d}{dx}(yx) = 3x^2$ is called an **exact equation**. It can effectively be solved by integrating both sides.

Task

Solve the equations (a) $\frac{dy}{dx} = 5x^4$ (b) $\frac{d}{dx}(x^3y) = 5x^4$

Your solution

(a) $y =$ (b) $y =$

Answer

(a) $y = x^5 + C$ (b) $x^3y = x^5 + C$ so that $y = x^2 + \frac{C}{x^3}$.

If we consider examples of this kind in a more general setting we obtain the following Key Point:

Key Point 2

The solution of the equation

$$\frac{d}{dx}(f(x)\cdot y) = g(x)$$

is

$$f(x)\cdot y = \int g(x)\,dx \qquad \text{or} \qquad y = \frac{1}{f(x)}\int g(x)\,dx$$

4. Solving exact equations

As we have seen, the differential equation $\dfrac{d}{dx}(yx) = 3x^2$ has solution $y = x^2 + C/x$. In the solution, x^2 is called the **definite part** and C/x is called the **indefinite part** (containing the arbitrary constant of integration). If we take the definite part of this solution, i.e. $y_{\mathsf{d}} = x^2$, then

$$\frac{d}{dx}(y_{\mathsf{d}}\cdot x) = \frac{d}{dx}(x^2\cdot x) = \frac{d}{dx}(x^3) = 3x^2.$$

Hence $y_{\mathsf{d}} = x^2$ is *a* solution of the differential equation.
Now if we take the indefinite part of the solution i.e. $y_{\mathsf{i}} = C/x$ then

$$\frac{d}{dx}(y_{\mathsf{i}}\cdot x) = \frac{d}{dx}\left(\frac{C}{x}\cdot x\right) = \frac{d}{dx}(C) = 0.$$

It is always the case that the general solution of an exact equation is in two parts: a definite part $y_{\mathsf{d}}(x)$ which is *a* solution of the differential equation and an indefinite part $y_{\mathsf{i}}(x)$ which satisfies a simpler version of the differential equation in which the right-hand side is zero.

Task

(a) Solve the equation

$$\frac{d}{dx}(y\cos x) = \cos x$$

(b) Verify that the indefinite part of the solution satisfies the equation

$$\frac{d}{dx}(y\cos x) = 0.$$

(a) Integrate both sides of the first differential equation:

Your solution

Answer

$$y\cos x = \int \cos x\,dx = \sin x + C \qquad \text{leading to} \qquad y = \tan x + C\sec x$$

(b) Substitute for y in the indefinite part (i.e. the part which contains the arbitrary constant) in the second differential equation:

Your solution

Answer

The indefinite part of the solution is $y_{\mathsf{i}} = C\sec x$ and so $y_{\mathsf{i}}\cos x = C$ and

$$\frac{d}{dx}(y_{\mathsf{i}}\cos x) = \frac{d}{dx}(C) = 0$$

5. Recognising an exact equation

The equation $\dfrac{d}{dx}(yx) = 3x^2$ is exact, as we have seen. If we expand the left-hand side of this equation (i.e. differentiate the product) we obtain

$$x\frac{dy}{dx} + y.$$

Hence the equation

$$x\frac{dy}{dx} + y = 3x^2$$

must be exact, but it is not so obvious that it is exact as in the original form. This leads to the following Key Point:

Key Point 3

The equation

$$f(x)\frac{dy}{dx} + y\ f'(x) = g(x)$$

is exact. It can be re-written as

$$\frac{d}{dx}(y\ f(x)) = g(x) \qquad \text{so that} \qquad y\ f(x) = \int g(x)\ dx$$

Example 4

Solve the equation

$$x^3\frac{dy}{dx} + 3x^2y = x$$

Solution

Comparing this equation with the form in Key Point 3 we see that $f(x) = x^3$ and $g(x) = x$. Hence the equation can be written

$$\frac{d}{dx}(yx^3) = x$$

which has solution

$$yx^3 = \int x\,dx = \tfrac{1}{2}x^2 + C.$$

Therefore

$$y = \frac{1}{2x} + \frac{C}{x^3}.$$

Task

Solve the equation $\sin x\,\dfrac{dy}{dx} + y\cos x = \cos x$.

Your solution

Answer

You should obtain $y = 1 + C\text{cosec}\,x$ since, here $f(x) = \sin x$ and $g(x) = \cos x$. Then

$$\frac{d}{dx}(y\sin x) = \cos x \qquad \text{and} \qquad y\sin x = \int \cos x\ dx = \sin x + C$$

Finally $y = 1 + C\,\text{cosec}\,x$.

Exercises

1. Solve the equation $\dfrac{d}{dx}(yx^2) = x^3$.

2. Solve the equation $\dfrac{d}{dx}(y\mathrm{e}^x) = \mathrm{e}^{2x}$ given the condition $y(0) = 2$.

3. Solve the equation $\mathrm{e}^{2x}\dfrac{dy}{dx} + 2\mathrm{e}^{2x}y = x^2$.

4. Show that the equation $x^2\dfrac{dy}{dx} + 2xy = x^3$ is exact and obtain its solution.

5. Show that the equation $x^2\dfrac{dy}{dx} + 3xy = x^3$ is not exact.
   Multiply the equation by x and show that the resulting equation **is** exact and obtain its solution.

Answers

1. $y = \dfrac{x^2}{4} + \dfrac{C}{x^2}$. 2. $y = \frac{1}{2}\mathrm{e}^x + \frac{3}{2}\mathrm{e}^{-x}$. 3. $y = \left(\frac{1}{3}x^3 + C\right)\mathrm{e}^{-2x}$. 4. $y = \dfrac{1}{4}x^2 + \dfrac{C}{x^2}$.
5. $y = \dfrac{1}{5}x^2 + \dfrac{C}{x^3}$.

6. The integrating factor

The equation

$$x^2\frac{dy}{dx} + 3x\ y = x^3$$

is not exact. However, if we multiply it by x we obtain the equation

$$x^3\frac{dy}{dx} + 3x^2y = x^4.$$

This can be re-written as

$$\frac{d}{dx}(x^3y) = x^4$$

which is an exact equation with solution

$$x^3y = \int x^4 dx$$

so $\qquad x^3y = \dfrac{1}{5}x^5 + C$

and hence

$$y = \frac{1}{5}x^2 + \frac{C}{x^3}.$$

The function by which we multiplied the given differential equation in order to make it exact is called an **integrating factor**. In this example the integrating factor is simply x.

Task

Which of the following differential equations can be made exact by multiplying by x^2?

(a) $\dfrac{dy}{dx} + \dfrac{2}{x}y = 4$ (b) $x\,\dfrac{dy}{dx} + 3y = x^2$ (c) $\dfrac{1}{x}\,\dfrac{dy}{dx} - \dfrac{1}{x^2}y = x$

(d) $\dfrac{1}{x}\,\dfrac{dy}{dx} + \dfrac{1}{x^2}y = 3.$

Where possible, write the exact equation in the form $\dfrac{d}{dx}(f(x)\ y) = g(x)$.

Your solution

Answer

(a) Yes. $x^2\dfrac{dy}{dx} + 2xy = 4x^2$ becomes $\dfrac{d}{dx}(x^2y) = 4x^2$.

(b) Yes. $x^3\dfrac{dy}{dx} + 3x^2y = x^4$ becomes $\dfrac{d}{dx}(x^3y) = x^4$.

(c) No. This equation is already exact as it can be written in the form $\dfrac{d}{dx}\left(\dfrac{1}{x}\ y\right) = x$.

(d) Yes. $x\dfrac{dy}{dx} + y = 3x^2$ becomes $\dfrac{d}{dx}(xy) = 3x^2$.

7. Finding the integrating factor for linear ODEs

The differential equation governing the current i in a circuit with inductance L and resistance R in series subject to a constant applied electromotive force $E\cos\omega t$, where E and ω are constants, is

$$L\,\frac{di}{dt} + Ri = E\cos\omega t \qquad (1)$$

This is an example of a linear differential equation in which i is the dependent variable and t is the independent variable. The general standard form of a linear first order differential equation is normally written with 'y' as the dependent variable and with 'x' as the independent variable and arranged so that the coefficient of $\dfrac{dy}{dx}$ is 1. That is, it takes the form:

$$\frac{dy}{dx} + f(x)\ y = g(x) \qquad (2)$$

in which $f(x)$ and $g(x)$ are functions of x.

Comparing (1) and (2), x is replaced by t and y by i to produce $\dfrac{di}{dt} + f(t)\ i = g(t)$. The function $f(t)$ is the coefficient of the dependent variable in the differential equation. We shall describe the method of finding the integrating factor for (1) and then generalise it to a linear differential equation written in standard form.

Step 1 Write the differential equation in standard form i.e. with the coefficient of the derivative equal to 1. Here we need to divide through by L:

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\cos\omega t.$$

Step 2 Integrate the coefficient of the dependent variable (that is, $f(t) = R/L$) with respect to the independent variable (that is, t), and ignoring the constant of integration

$$\int \frac{R}{L}\,dt = \frac{R}{L}\,t.$$

Step 3 Take the exponential of the function obtained in Step 2.

This is the integrating factor (I.F.)

$$\text{I.F.} = \mathrm{e}^{Rt/L}.$$

This leads to the following Key Point on integrating factors:

Key Point 4

The linear differential equation (written in standard form):

$$\frac{dy}{dx} + f(x)y = g(x) \qquad \text{has an integrating factor} \qquad \text{I.F.} = \exp\left[\int f(x)dx\right]$$

Task

Find the integrating factors for the equations

(a) $x\,\dfrac{dy}{dx} + 2x\ y = x\mathrm{e}^{-2x}$ (b) $t\,\dfrac{di}{dt} + 2t\,i = t\mathrm{e}^{-2t}$ (c) $\dfrac{dy}{dx} - (\tan x)y = 1.$

Your solution

Answer

(a) **Step 1** Divide by x to obtain $\dfrac{dy}{dx} + 2y = \mathrm{e}^{-2x}$

Step 2 The coefficient of the independent variable is 2 hence $\displaystyle\int 2\ dx = 2x$

Step 3 I.F. $= \mathrm{e}^{2x}$

(b) The only difference from (a) is that i replaces y and t replaces x. Hence I.F. $= \mathrm{e}^{2t}$.

(c) **Step 1** This is already in the standard form.

Step 2 $\displaystyle\int -\tan x\ dx = \int \frac{-\sin x}{\cos x}\ dx = \ln\cos x.$

Step 3 I.F. $= \mathrm{e}^{\ln\cos x} = \cos x$

8. Solving equations via the integrating factor

Having found the integrating factor for a linear equation we now proceed to solve the equation. Returning to the differential equation, written in standard form:

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\cos\omega t$$

for which the integrating factor is

$$\mathrm{e}^{Rt/L}$$

we multiply the equation by the integrating factor to obtain

$$\mathrm{e}^{Rt/L}\ \frac{di}{dt} + \frac{R}{L}\ \mathrm{e}^{Rt/L}\ i = \frac{E}{L}\ \mathrm{e}^{Rt/L}\cos\omega t$$

At this stage the left-hand side of this equation can **always** be simplified as follows:

$$\frac{d}{dt}(\mathrm{e}^{Rt/L}\ i) = \frac{E}{L}\ \mathrm{e}^{Rt/L}\cos\omega t.$$

Now this is in the form of an exact differential equation and so we can integrate both sides to obtain the solution:

$$\mathrm{e}^{Rt/L}\ i = \frac{E}{L}\int \mathrm{e}^{Rt/L}\cos\omega t\ dt.$$

All that remains is to complete the integral on the right-hand side. Using the method of integration by parts we find

$$\int \mathrm{e}^{Rt/L}\cos\omega t\ dt = \frac{L}{L^2\omega^2 + R^2}\left[\omega L\ \sin\omega t + R\cos\omega t\right]\mathrm{e}^{Rt/L}$$

Hence

$$\mathrm{e}^{Rt/L}\ i = \frac{E}{L^2\omega^2 + R^2}\left[\omega L\sin\omega t + R\cos\omega t\right]\ \mathrm{e}^{Rt/L} + C.$$

Finally

$$i = \frac{E}{L^2\omega^2 + R^2}\left[\omega L\sin\omega t + R\cos\omega t\right] + C\ \mathrm{e}^{-Rt/L}.$$

is the solution to the original differential equation (1). Note that, as we should expect for the solution to a first order differential equation, it contains a single arbitrary constant C.

Task

Using the integrating factors found earlier in the Task on pages 22-23, find the general solutions to the differential equations

(a) $x^2\,\dfrac{dy}{dx} + 2x^2y = x^2\text{e}^{-2x}$ (b) $t^2\,\dfrac{di}{dt} + 2t^2i = t^2\text{e}^{-2t}$ (c) $\dfrac{dy}{dx} - (\tan x)y = 1.$

Your solution

Answer

(a) The standard form is $\dfrac{dy}{dx} + 2y = \text{e}^{-2x}$ for which the integrating factor is e^{2x}.

$$\begin{aligned}
\text{e}^{2x}\frac{dy}{dx} + 2\text{e}^{2x}\,y &= 1 \\
\text{i.e.}\qquad \frac{d}{dx}\,(\text{e}^{2x}\,y) &= 1 \qquad \text{so that} \qquad \text{e}^{2x}y = x + C \\
\text{leading to}\qquad y &= (x+C)\text{e}^{-2x}
\end{aligned}$$

(b) The general solution is $i = (t+C)\text{e}^{-2t}$ as this problem is the same as (a) with different variables.

(c) The equation is in standard form and the integrating factor is $\cos x$.

$$\begin{aligned}
\text{then}\qquad \frac{d}{dx}(\cos x\;y) &= \cos x \qquad \text{so that} \qquad \cos x\;y = \int \cos x\,dx = \sin x + C \\
\text{giving}\qquad y &= \tan x + C\sec x
\end{aligned}$$

Engineering Example 1

An RC circuit with a single frequency input

Introduction

The components in RC circuits containing resistance, inductance and capacitance can be chosen so that the circuit filters out certain frequencies from the input. A particular kind of filter circuit consists of a resistor and capacitor in series and acts as a high cut (or low pass) filter. The high cut frequency is defined to be the frequency at which the magnitude of the voltage across the capacitor (the output voltage) is $1/\sqrt{2}$ of the magnitude of the input voltage.

Problem in words

Calculate the high cut frequency for an RC circuit is subjected to a single frequency input of angular frequency ω and magnitude v_i.

(a) Find the steady state solution of the equation

$$R\frac{dq}{dt} + \frac{q}{C} = v_i e^{j\omega t}$$

and hence find the magnitude of

(i) the voltage across the capacitor $v_c = \dfrac{q}{C}$

(ii) the voltage across the resistor $v_R = R\,\dfrac{dq}{dt}$

(b) Using the impedance method of HELM 12.6 confirm your results to part (a) by calculating

(i) the voltage across the capacitor v_c

(ii) the voltage across the resistor v_R in response to a single frequency of angular frequency ω and magnitude v_i.

(c) For the case where $R = 1\ k\Omega$ and $C = 1\ \mu F$, find the ratio $\dfrac{|v_c|}{|v_i|}$ and complete the table below

ω	10	10^2	10^3	10^4	10^5	10^6
$\dfrac{\lvert v_c\rvert}{\lvert v_i\rvert}$						

(d) Explain why the table results show that a RC circuit acts as a high-cut filter and find the value of the high-cut frequency, defined as $f_{hc} = \omega_{hc}/2\pi$, such that $\dfrac{|v_c|}{|v_i|} = \dfrac{1}{\sqrt{2}}$.

Mathematical statement of the problem

We need to find a particular solution to the differential equation $R\,\dfrac{dq}{dt} + \dfrac{q}{C} = v_i e^{j\omega t}$.
This will give us the steady state solution for the charge q. Using this we can find $v_c = \dfrac{q}{C}$ and $v_R = R\,\dfrac{dq}{dt}$. These should give the same result as the values calculated by considering the impedances in the circuit. Finally we can calculate $\dfrac{|v_c|}{|v_i|}$ and fill in the table of values as required and find the high-cut frequency from $\dfrac{|v_c|}{|v_i|} = \dfrac{1}{\sqrt{2}}$ and $f_{hc} = \omega_{hc}/2\pi$.

Mathematical solution

(a) To find a particular solution, we try a function of the form $q = c_0 e^{j\omega t}$ which means that

$$\frac{dq}{dt} = j\omega c_0 e^{j\omega t}.$$

Substituting into $R\,\dfrac{dq}{dt} + \dfrac{q}{C} = v_i e^{j\omega t}$ we get

$$Rj\omega c_0 e^{j\omega t} + \frac{c_0 e^{j\omega t}}{C} = v_i e^{j\omega t} \quad \Rightarrow \quad Rj\omega c_0 + \frac{c_0}{C} = v_i$$

$$\Rightarrow \quad c_0 = \frac{v_i}{Rj\omega + \frac{1}{C}} = \frac{Cv_i}{RCj\omega + 1} \quad \Rightarrow \quad q = \frac{Cv_i}{RCj\omega + 1}\, e^{j\omega t}$$

Thus

(i) $v_c = \dfrac{q}{C} = \dfrac{v_i}{RCj\omega + 1}\, e^{j\omega t}$ and (ii) $v_R = \dfrac{dq}{dt} = \dfrac{RCv_i j\omega}{RCj\omega + 1}\, e^{j\omega t}$

(b) We use the impedance to determine the voltage across each of the elements. The applied voltage is a single frequency of angular frequency ω and magnitude v_i such that $V = v_i e^{j\omega t}$.

For an RC circuit, the impedance of the circuit is $Z = Z_R + Z_c$ where Z_R is the impedance of the resistor R and Z_c is the impedance of the capacitor $Z_c = -\dfrac{j}{\omega C}$.

Therefore $Z = R - \dfrac{j}{\omega C}$.

The current can be found using $v = Zi$ giving

$$v_i e^{j\omega t} = \left(R - \frac{j}{\omega C}\right) i \quad \Rightarrow \quad i = \frac{v_i e^{j\omega t}}{R - \frac{j}{\omega C}}$$

We can now use $v_c = z_c i$ and $v_R = z_R i$ giving

(i) $v_c = \dfrac{q}{C} = -\dfrac{j}{\omega C} \times \dfrac{v_i}{R - \frac{j}{\omega C}}\, e^{j\omega t} = \dfrac{v_i}{RCj\omega + 1}\, e^{j\omega t}$

(ii) $v_R = \dfrac{Rv_i}{R - \frac{j}{\omega C}}\, e^{j\omega t} = \dfrac{RCv_i j\omega}{RCj\omega + 1}\, e^{j\omega t}$

which confirms the result in part (a) found by solving the differential equation.

(c) When $R = 1000\ \Omega$ and $C = 10^{-6} F$

$$v_c = \frac{v_i}{RCj\omega + 1}\, e^{j\omega t} = \frac{v_i}{10^{-3} j\omega + 1}\, e^{j\omega t}$$

So $\dfrac{|v_c|}{|v_i|} = \left|\dfrac{1}{10^{-3}j\omega + 1}\right| \; |e^{j\omega t}| = \left|\dfrac{1}{10^{-3}j\omega + 1}\right| = \dfrac{1}{\sqrt{10^{-6}\omega^2 + 1}}$

Table 1: Values of $\left|\dfrac{v_c}{v_i}\right|$ for a range of values of ω

ω	10	10^2	10^3	10^4	10^5	10^6
$\dfrac{\lvert v_c\rvert}{\lvert v_i\rvert}$	0.99995	0.995	0.707	0.00995	0.0099995	0.001

(d) Table 1 shows that a RC circuit can be used as a high-cut filter because for low values of ω, $\dfrac{|v_c|}{|v_i|}$ is approximately 1 and for high values of ω, $\dfrac{|v_c|}{|v_i|}$ is approximately 0. So the circuit will filter out high frequency values.

$$\frac{|v_c|}{|v_i|} = \frac{1}{\sqrt{2}} \text{ when } \frac{1}{\sqrt{10^{-6}\omega^2 + 1}} = \frac{1}{\sqrt{2}} \quad \Leftrightarrow \quad 10^{-6}\omega^2 + 1 = 2 \Leftrightarrow 10^{-6}\omega^2 = 1 \Leftrightarrow \omega^2 = 10^6$$

As we are considering ω to be a positive frequency, $\omega = 1000.$

So $f_{hc} = \dfrac{\omega_{hc}}{2\pi} = \dfrac{1000}{2\pi} \approx 159$ Hz.

Interpretation

We have shown that for an RC circuit finding the steady state solution of the differential equation with a single frequency input voltage yields the same result for $\dfrac{|v_c|}{|v_i|}$ and $\dfrac{|v_R|}{|v_i|}$ as found by working with the complex impedances for the circuit.

An RC circuit can be used as a high-cut filter and in the case where $R = 1\ k\Omega, C = 1\ \mu F$ we found the high-cut frequency to be at approximately 159 Hz.

This means that the circuit will pass frequencies less than this value and remove frequencies greater than this value.

Exercises

1. Solve the equation $x^2\dfrac{dy}{dx} + x\ y = 1$.
2. Find the solution of the equation $x\dfrac{dy}{dx} - y = x$ subject to the condition $y(1) = 2$.
3. Find the general solution of the equation $\dfrac{dy}{dt} + (\tan t)\ y = \cos t$.
4. Solve the equation $\dfrac{dy}{dt} + (\cot t)\ y = \sin t$.
5. The temperature θ (measured in degrees) of a body immersed in an atmosphere of varying temperature is given by $\dfrac{d\theta}{dt} + 0.1\theta = 5 - 2.5t$. Find the temperature at time t if $\theta = 60°$C when $t = 0$.
6. In an LR circuit with applied voltage $E = 10(1 - \mathrm{e}^{-0.1t})$ the current i is given by

   $$L\frac{di}{dt} + Ri = 10(1 - \mathrm{e}^{-0.1t}).$$

   If the initial current is i_0 find i subsequently.

Answers

1. $y = \dfrac{1}{x}\ln x + \dfrac{C}{x}$
2. $y = x\ln x + 2x$
3. $y = (t + C)\cos t$
4. $y = \left(\frac{1}{2}\ t - \frac{1}{4}\sin 2t + C\right)\mathrm{cosec}\, t$
5. $\theta = 300 - 25t - 240\mathrm{e}^{-0.1t}$
6. $i = \dfrac{10}{R} - \left(\dfrac{100}{10R - L}\right)\mathrm{e}^{-0.1t} + \left[i_0 + \dfrac{10L}{R(10R - L)}\ \right]\mathrm{e}^{-Rt/L}$

Second Order Differential Equations 19.3

Introduction

In this Section we start to learn how to solve second order differential equations of a particular type: those that are linear and have constant coefficients. Such equations are used widely in the modelling of physical phenomena, for example, in the analysis of vibrating systems and the analysis of electrical circuits.

The solution of these equations is achieved in stages. The first stage is to find what is called a 'complementary function'. The second stage is to find a 'particular integral'. Finally, the complementary function and the particular integral are combined to form the general solution.

Prerequisites

Before starting this Section you should ...

- understand what is meant by a differential equation
- understand complex numbers (HELM 10)

Learning Outcomes

On completion you should be able to ...

- recognise a linear, constant coefficient equation
- understand what is meant by the terms 'auxiliary equation' and 'complementary function'
- find the complementary function when the auxiliary equation has real, equal or complex roots

1. Constant coefficient second order linear ODEs

We now proceed to study those second order linear equations which have constant coefficients. The general form of such an equation is:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \qquad (3)$$

where a, b, c are constants. The **homogeneous** form of (3) is the case when $f(x) \equiv 0$:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (4)$$

To find the general solution of (3), it is first necessary to solve (4). The general solution of (4) is called the **complementary function** and will always contain two arbitrary constants. We will denote this solution by y_{cf}.

The technique for finding the complementary function is described in this Section.

Task

State which of the following are constant coefficient equations.
State which are homogeneous.

(a) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = \mathrm{e}^{-2x}$ (b) $x\dfrac{d^2y}{dx^2} + 2y = 0$

(c) $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + 7x = 0$ (d) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$

Your solution

(a)

(b)

(c)

(d)

Answer

(a) is constant coefficient and is not homogeneous.

(b) is homogeneous but not constant coefficient as the coefficient of $\dfrac{d^2y}{dx^2}$ is x, a variable.

(c) is constant coefficient and homogeneous. In this example the dependent variable is x.

(d) is constant coefficient and homogeneous.

Note: A complementary function is the general solution of a homogeneous, linear differential equation.

2. Finding the complementary function

To find the complementary function we must make use of the following property.

If $y_1(x)$ and $y_2(x)$ are any two (linearly independent) solutions of a linear, homogeneous second order differential equation then the general solution $y_{\text{cf}}(x)$, is

$$y_{\text{cf}}(x) = Ay_1(x) + By_2(x)$$

where $A,\ B$ are constants.

We see that the second order linear ordinary differential equation has two arbitrary constants in its **general solution**. The functions $y_1(x)$ and $y_2(x)$ are **linearly independent** if one is not a multiple of the other.

Example 5

Verify that $y_1 = \mathrm{e}^{4x}$ and $y_2 = \mathrm{e}^{2x}$ both satisfy the constant coefficient linear homogeneous equation:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$$

Write down the general solution of this equation.

Solution

When $y_1 = \mathrm{e}^{4x}$, differentiation yields:

$$\frac{dy_1}{dx} = 4\mathrm{e}^{4x} \qquad \text{and} \qquad \frac{d^2y_1}{dx^2} = 16\mathrm{e}^{4x}$$

Substitution into the left-hand side of the ODE gives $16\mathrm{e}^{4x} - 6(4\mathrm{e}^{4x}) + 8\mathrm{e}^{4x}$, which equals 0, so that $y_1 = \mathrm{e}^{4x}$ is indeed a solution.

Similarly if $y_2 = \mathrm{e}^{2x}$, then

$$\frac{dy_2}{dx} = 2\mathrm{e}^{2x} \qquad \text{and} \qquad \frac{d^2y_2}{dx^2} = 4\mathrm{e}^{2x}.$$

Substitution into the left-hand side of the ODE gives $4\mathrm{e}^{2x} - 6(2\mathrm{e}^{2x}) + 8\mathrm{e}^{2x}$, which equals 0, so that $y_2 = \mathrm{e}^{2x}$ is also a solution of equation the ODE. Now e^{2x} and e^{4x} are linearly independent functions, so, from the property stated above we have:

$$y_{\text{cf}}(x) = A\mathrm{e}^{4x} + B\mathrm{e}^{2x} \quad \text{is the general solution of the ODE.}$$

Example 6

Find values of k so that $y = \mathrm{e}^{kx}$ is a solution of:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

Hence state the general solution.

Solution

As suggested we try a solution of the form $y = \mathrm{e}^{kx}$. Differentiating we find

$$\frac{dy}{dx} = k\mathrm{e}^{kx} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = k^2\mathrm{e}^{kx}.$$

Substitution into the given equation yields:

$$k^2\mathrm{e}^{kx} - k\mathrm{e}^{kx} - 6\mathrm{e}^{kx} = 0 \qquad \text{that is} \qquad (k^2 - k - 6)\mathrm{e}^{kx} = 0$$

The only way this equation can be satisfied for all values of x is if

$$k^2 - k - 6 = 0$$

that is, $(k-3)(k+2) = 0$ so that $k = 3$ or $k = -2$. That is to say, if $y = \mathrm{e}^{kx}$ is to be a solution of the differential equation, k must be either 3 or -2. We therefore have found two solutions:

$$y_1(x) = \mathrm{e}^{3x} \qquad \text{and} \qquad y_2(x) = \mathrm{e}^{-2x}$$

These are linearly independent and therefore the general solution is

$$y_{\mathsf{cf}}(x) = A\mathrm{e}^{3x} + B\mathrm{e}^{-2x}$$

The equation $k^2 - k - 6 = 0$ for determining k is called the **auxiliary equation**.

Task

By substituting $y = \mathrm{e}^{kx}$, find values of k so that y is a solution of

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Hence, write down two solutions, and the general solution of this equation.

First find the auxiliary equation:

Your solution

Answer

$k^2 - 3k + 2 = 0$

Now solve the auxiliary equation and write down the general solution:

Your solution

Answer
The auxiliary equation can be factorised as $(k-1)(k-2) = 0$ and so the required values of k are 1 and 2. The two solutions are $y = \mathrm{e}^x$ and $y = \mathrm{e}^{2x}$. The general solution is

$$y_{\mathsf{cf}}(x) = A\mathrm{e}^x + B\mathrm{e}^{2x}$$

Example 7

Find the auxiliary equation of the differential equation:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Solution

We try a solution of the form $y = \mathrm{e}^{kx}$ so that

$$\frac{dy}{dx} = k\mathrm{e}^{kx} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = k^2\mathrm{e}^{kx}.$$

Substitution into the given differential equation yields:

$$ak^2\mathrm{e}^{kx} + bk\mathrm{e}^{kx} + c\mathrm{e}^{kx} = 0 \qquad \text{that is} \qquad (ak^2 + bk + c)\mathrm{e}^{kx} = 0$$

Since this equation is to be satisfied for all values of x, then

$$ak^2 + bk + c = 0$$

is the required auxiliary equation.

Key Point 5

The **auxiliary equation** of $\quad a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0 \quad$ is $\quad ak^2 + bk + c = 0\;$ where $\;y = \mathrm{e}^{kx}$

Task

Write down, but do not solve, the auxiliary equations of the following:

(a) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0,$ (b) $2\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} - 3y = 0$

(c) $4\dfrac{d^2y}{dx^2} + 7y = 0,$ (d) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$

Your solution

(a)

(b)

(c)

(d)

Answer

(a) $k^2 + k + 1 = 0$ (b) $2k^2 + 7k - 3 = 0$ (c) $4k^2 + 7 = 0$ (d) $k^2 + k = 0$

Solving the auxiliary equation gives the values of k which we need to find the complementary function. Clearly the nature of the roots will depend upon the values of $a,\ b$ and c.

Case 1 If $b^2 > 4ac$ the roots will be real and distinct. The two values of k thus obtained, k_1 and k_2, will allow us to write down two independent solutions: $y_1(x) = \mathrm{e}^{k_1x}$ and $y_2(x) = \mathrm{e}^{k_2x}$, and so the general solution of the differential equation will be:

$$y(x) = A\mathrm{e}^{k_1x} + B\mathrm{e}^{k_2x}$$

Key Point 6

If the auxiliary equation has real, distinct roots k_1 and k_2, the **complementary function** will be:

$$y_{\text{cf}}(x) = A\mathrm{e}^{k_1x} + B\mathrm{e}^{k_2x}$$

Case 2 On the other hand, if $b^2 = 4ac$ the two roots of the auxiliary equation will be equal and this method will therefore only yield one independent solution. In this case, special treatment is required.

Case 3 If $b^2 < 4ac$ the two roots of the auxiliary equation will be complex, that is, k_1 and k_2 will be complex numbers. The procedure for dealing with such cases will become apparent in the following examples.

Example 8

Find the general solution of: $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 10y = 0$

Solution

By letting $y = \mathrm{e}^{kx}$, so that $\dfrac{dy}{dx} = k\mathrm{e}^{kx}$ and $\dfrac{d^2y}{dx^2} = k^2\mathrm{e}^{kx}$

the auxiliary equation is found to be: $k^2 + 3k - 10 = 0$ and so $(k-2)(k+5) = 0$

so that $k = 2$ and $k = -5$. Thus there exist two solutions: $y_1 = \mathrm{e}^{2x}$ and $y_2 = \mathrm{e}^{-5x}$.

We can write the general solution as: $y = A\mathrm{e}^{2x} + B\mathrm{e}^{-5x}$

Example 9

Find the general solution of: $\dfrac{d^2y}{dx^2} + 4y = 0$

Solution

As before, let $y = \mathrm{e}^{kx}$ so that $\dfrac{dy}{dx} = k\mathrm{e}^{kx}$ and $\dfrac{d^2y}{dx^2} = k^2\mathrm{e}^{kx}$.

The auxiliary equation is easily found to be: $k^2 + 4 = 0$ that is, $k^2 = -4$ so that $k = \pm 2\mathrm{i}$, that is, we have complex roots. The two independent solutions of the equation are thus

$$y_1(x) = \mathrm{e}^{2\mathrm{i}x} \qquad y_2(x) = \mathrm{e}^{-2\mathrm{i}x}$$

so that the general solution can be written in the form $y(x) = A\mathrm{e}^{2\mathrm{i}x} + B\mathrm{e}^{-2\mathrm{i}x}$.

However, in cases such as this, it is usual to rewrite the solution in the following way.

Recall that Euler's relations give: $\mathrm{e}^{2\mathrm{i}x} = \cos 2x + \mathrm{i}\sin 2x$ and $\mathrm{e}^{-2\mathrm{i}x} = \cos 2x - \mathrm{i}\sin 2x$

so that $y(x) = A(\cos 2x + \mathrm{i}\sin 2x) + B(\cos 2x - \mathrm{i}\sin 2x)$.

If we now relabel the constants such that $A + B = C$ and $A\mathrm{i} - B\mathrm{i} = D$ we can write the general solution in the form:

$$y(x) = C\cos 2x + D\sin 2x$$

Note: In Example 8 we have expressed the solution as $y = \ldots$ whereas in Example 9 we have expressed it as $y(x) = \ldots$. Either will do.

Example 10

Given $ay'' + by' + cy = 0$, write down the auxiliary equation. If the roots of the auxiliary equation are complex (one root will always be the complex conjugate of the other) and are denoted by $k_1 = \alpha + \beta\text{i}$ and $k_2 = \alpha - \beta\text{i}$ show that the general solution is:

$$y(x) = \text{e}^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

Solution

Substitution of $y = \text{e}^{kx}$ into the differential equation yields $(ak^2+bk+c)\text{e}^{kx} = 0$ and so the auxiliary equation is:

$$ak^2 + bk + c = 0$$

If $k_1 = \alpha + \beta\text{i},\ k_2 = \alpha - \beta\text{i}$ then the general solution is

$$y = C\text{e}^{(\alpha+\beta\text{i})x} + D\text{e}^{(\alpha-\beta\text{i})x}$$

where C and D are arbitrary constants.

Using the laws of indices this is rewritten as:

$$y = C\text{e}^{\alpha x}\text{e}^{\beta\text{i}x} + D\text{e}^{\alpha x}\text{e}^{-\beta\text{i}x} = \text{e}^{\alpha x}(C\text{e}^{\beta\text{i}x} + D\text{e}^{-\beta\text{i}x})$$

Then, using Euler's relations, we obtain:

$$\begin{aligned} y &= \text{e}^{\alpha x}(C\cos\beta x + C\text{i}\sin\beta x + D\cos\beta x - D\text{i}\sin\beta x) \\ &= \text{e}^{\alpha x}\{(C+D)\cos\beta x + (C\text{i} - D\text{i})\sin\beta x\} \end{aligned}$$

Writing $A = C + D$ and $B = C\text{i} - D\text{i}$, we find the required solution:

$$y = \text{e}^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

Key Point 7

If the auxiliary equation has complex roots, $\alpha + \beta\text{i}$ and $\alpha - \beta\text{i}$, then the **complementary function** is:

$$y_{\text{cf}} = \text{e}^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

Task Find the general solution of $y'' + 2y' + 4y = 0$.

Write down the auxiliary equation:

Your solution

Answer

$k^2 + 2k + 4 = 0$

Find the complex roots of the auxiliary equation:

Your solution

Answer

$k = -1 \pm \sqrt{3}\mathrm{i}$

Using Key Point 7 with $\alpha = -1$ and $\beta = \sqrt{3}$ write down the general solution:

Your solution

Answer

$y = \mathrm{e}^{-x}(A \cos \sqrt{3}x + B \sin \sqrt{3}x)$

Key Point 8

If the auxiliary equation has two equal roots, k, the **complementary function** is:

$$y_{\mathsf{cf}} = (A + Bx)\mathrm{e}^{kx}$$

Example 11

The auxiliary equation of $ay'' + by' + cy = 0$ is $ak^2 + bk + c = 0$. Suppose this equation has equal roots $k = k_1$ and $k = k_1$. Verify that $y = xe^{k_1 x}$ is a solution of the differential equation.

Solution

We have: $\quad y = xe^{k_1 x} \qquad y' = e^{k_1 x}(1 + k_1 x) \qquad y'' = e^{k_1 x}(k_1^2 x + 2k_1)$

Substitution into the left-hand side of the differential equation yields:

$$e^{k_1 x}\{a(k_1^2 x + 2k_1) + b(1 + k_1 x) + cx\} = e^{k_1 x}\{(ak_1^2 + bk_1 + c)x + 2ak_1 + b\}$$

But $ak_1^2 + bk_1 + c = 0$ since k_1 satisfies the auxiliary equation. Also,

$$k_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

but since the roots are equal, then $b^2 - 4ac = 0$ hence $k_1 = -b/2a$. So $2ak_1 + b = 0$. Hence $e^{k_1 x}\{(ak_1^2 + bk_1 + c)x + 2ak_1 + b\} = e^{k_1 x}\{(0)x + 0\} = 0$. We conclude that $y = xe^{k_1 x}$ is a solution of $ay'' + by' + cy = 0$ when the roots of the auxiliary equation are equal. This illustrates Key Point 8.

Example 12

Obtain the general solution of the equation: $\quad \dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 16y = 0.$

Solution

As before, a trial solution of the form $y = e^{kx}$ yields an auxiliary equation $k^2 + 8k + 16 = 0$. This equation factorizes so that $(k + 4)(k + 4) = 0$ and we obtain equal roots, that is, $k = -4$ (twice). If we proceed as before, writing $y_1(x) = e^{-4x}$ $y_2(x) = e^{-4x}$, it is clear that the two solutions are not independent. We need to find a second independent solution. Using the result summarised in Key Point 8, we conclude that the second independent solution is $y_2 = xe^{-4x}$. The general solution is then:

$$y(x) = (A + Bx)e^{-4x}$$

Exercises

1. Obtain the general solutions, that is, the complementary functions, of the following equations:

(a) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$ (b) $\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} + 6y = 0$ (c) $\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + 6x = 0$

(d) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 0$ (e) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$ (f) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + 8y = 0$

(g) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$ (h) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + 5y = 0$ (i) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 0$

(j) $\dfrac{d^2y}{dx^2} + 9y = 0$ (k) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = 0$ (l) $\dfrac{d^2x}{dt^2} - 16x = 0$

2. Find the auxiliary equation for the differential equation $L\dfrac{d^2i}{dt^2} + R\dfrac{di}{dt} + \dfrac{1}{C}i = 0$

   Hence write down the complementary function.

3. Find the complementary function of the equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$

Answers

1. (a) $y = A\mathrm{e}^{x} + B\mathrm{e}^{2x}$
   (b) $y = A\mathrm{e}^{-x} + B\mathrm{e}^{-6x}$
   (c) $x = A\mathrm{e}^{-2t} + B\mathrm{e}^{-3t}$
   (d) $y = A\mathrm{e}^{-t} + Bt\mathrm{e}^{-t}$
   (e) $y = A\mathrm{e}^{2x} + Bx\mathrm{e}^{2x}$
   (f) $y = \mathrm{e}^{-0.5t}(A\cos 2.78t + B\sin 2.78t)$
   (g) $y = A\mathrm{e}^{x} + Bx\mathrm{e}^{x}$
   (h) $x = \mathrm{e}^{-0.5t}(A\cos 2.18t + B\sin 2.18t)$
   (i) $y = A\mathrm{e}^{-2x} + B\mathrm{e}^{x}$
   (j) $y = A\cos 3x + B\sin 3x$
   (k) $y = A + B\mathrm{e}^{2x}$
   (l) $x = A\mathrm{e}^{4t} + B\mathrm{e}^{-4t}$

2. $Lk^2 + Rk + \dfrac{1}{C} = 0$ $\quad i(t) = A\mathrm{e}^{k_1 t} + B\mathrm{e}^{k_2 t}$ $\quad k_1, k_2 = \dfrac{1}{2L}\left(-R \pm \sqrt{\dfrac{R^2C - 4L}{C}}\right)$

3. $\mathrm{e}^{-x/2}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$

3. The particular integral

Given a second order ODE

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = f(x),$$

a **particular integral** is any function, $y_{\mathsf{p}}(x)$, which satisfies the equation. That is, any function which when substituted into the left-hand side, results in the expression on the right-hand side.

Task

Show that

$$y = -\tfrac{1}{4}\mathrm{e}^{2x}$$

is a particular integral of

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = \mathrm{e}^{2x} \qquad (1)$$

Starting with $y = -\frac{1}{4}\mathrm{e}^{2x}$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$:

Your solution

Answer

$\dfrac{dy}{dx} = -\frac{1}{2}\mathrm{e}^{2x}, \quad \dfrac{d^2y}{dx^2} = -\mathrm{e}^{2x}$

Now substitute these into the ODE and simplify to check it satisfies the equation:

Your solution

Answer

Substitution yields $-\mathrm{e}^{2x} - \left(-\frac{1}{2}\mathrm{e}^{2x}\right) - 6\left(-\frac{1}{4}\mathrm{e}^{2x}\right)$ which simplifies to e^{2x}, the same as the right-hand side.

Therefore $y = -\frac{1}{4}\mathrm{e}^{2x}$ is a particular integral and we write (attaching a subscript p):

$$y_{\mathsf{p}}(x) = -\tfrac{1}{4}\mathrm{e}^{2x}$$

Task

State what is meant by a particular integral.

Your solution

Answer

A particular integral is **any** solution of a differential equation.

4. Finding a particular integral

In the previous subsection we explained what is meant by a particular integral. Now we look at a simple method to find a particular integral. In fact our method is rather crude. It involves trial and error and educated guesswork. We try solutions which are of the same general form as the $f(x)$ on the right-hand side.

Example 13

Find a particular integral of the equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = \mathrm{e}^{2x}$$

Solution

We shall attempt to find a solution of the inhomogeneous problem by trying a function of the same form as that on the right-hand side of the ODE. In particular, let us try $y(x) = A\mathrm{e}^{2x}$, where A is a constant that we shall now determine. If $y(x) = A\mathrm{e}^{2x}$ then

$$\frac{dy}{dx} = 2A\mathrm{e}^{2x} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = 4A\mathrm{e}^{2x}.$$

Substitution in the ODE gives:

$$4A\mathrm{e}^{2x} - 2A\mathrm{e}^{2x} - 6A\mathrm{e}^{2x} = \mathrm{e}^{2x}$$

that is,

$$-4A\mathrm{e}^{2x} = \mathrm{e}^{2x}$$

To ensure that y is a solution, we require $-4A = 1$, that is, $A = -\frac{1}{4}$.

Therefore the particular integral is $y_{\mathsf{p}}(x) = -\frac{1}{4}\mathrm{e}^{2x}$.

In Example 13 we chose a trial solution $A\mathrm{e}^{2x}$ of the same form as the ODE's right-hand side. Table 2 provides a summary of the trial solutions which should be tried for various forms of the right-hand side.

Table 2: Trial solutions to find the particular integral

	$f(x)$	Trial solution
(1)	constant term c	constant term k
(2)	linear, $ax + b$	$Ax + B$
(3)	polynomial in x of degree r: $ax^r + \cdots + bx + c$	polynomial in x of degree r: $Ax^r + \cdots + Bx + k$
(4)	$a \cos kx$	$A \cos kx + B \sin kx$
(5)	$a \sin kx$	$A \cos kx + B \sin kx$
(6)	$a\text{e}^{kx}$	$A\text{e}^{kx}$
(7)	$a\text{e}^{-kx}$	$A\text{e}^{-kx}$

Task

By trying a solution of the form $y = \alpha\text{e}^{-x}$ find a particular integral of the equation
$\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 3\text{e}^{-x}$

Substitute $y = \alpha\text{e}^{-x}$ into the given equation to find α, and hence find the particular integral:

Your solution

Answer

$\alpha = -\frac{3}{2}$; $\quad y_{\text{p}}(x) = -\frac{3}{2}\text{e}^{-x}$

Example 14

Obtain a particular integral of the equation: $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = x.$

Solution

In Example 13 and the last Task, we found that a fruitful approach for a first order ODE was to assume a solution in the same form as that on the right-hand side. Suppose we assume a solution $y(x) = \alpha x$ and proceed to determine α. This approach will actually fail, but let us see why. If $y(x) = \alpha x$ then $\dfrac{dy}{dx} = \alpha$ and $\dfrac{d^2y}{dx^2} = 0$. Substitution into the differential equation yields $0 - 6\alpha + 8\alpha x = x$ and α.

Comparing coefficients of x:

$$8\alpha x = x \qquad \text{so} \quad \alpha = \frac{1}{8}$$

Comparing constants: $\quad -6\alpha = 0 \qquad$ so $\quad \alpha = 0$

We have a contradiction! Clearly a particular integral of the form αx is not possible. The problem arises because differentiation of the term αx produces constant terms which are unbalanced on the right-hand side. So, we try a solution of the form $y(x) = \alpha x + \beta$ with α, β constants. This is consistent with the recommendation in Table 2 on page 43. Proceeding as before $\dfrac{dy}{dx} = \alpha$, $\dfrac{d^2y}{dx^2} = 0$. Substitution in the differential equation now gives:

$$0 - 6\alpha + 8(\alpha x + \beta) = x$$

Equating coefficients of x and then equating constant terms we find:

$$8\alpha = 1 \qquad (1)$$

$$-6\alpha + 8\beta = 0 \qquad (2)$$

From (1), $\alpha = \frac{1}{8}$ and then from (2) $\beta = \frac{3}{32}$.

The required particular integral is $y_{\mathsf{p}}(x) = \frac{1}{8}x + \frac{3}{32}$.

Task

Find a particular integral for the equation:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 3\cos x$$

First decide on an appropriate form for the trial solution, referring to Table 2 (page 43) if necessary:

Your solution

Answer

From Table 2, $y = A\cos x + B\sin x$, A and B constants.

Now find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ and substitute into the differential equation:

Your solution

Answer

Differentiating, we find:

$$\frac{dy}{dx} = -A\sin x + B\cos x \qquad \frac{d^2y}{dx^2} = -A\cos x - B\sin x$$

Substitution into the differential equation gives:

$$(-A\cos x - B\sin x) - 6(-A\sin x + B\cos x) + 8(A\cos x + B\sin x) = 3\cos x$$

Equate coefficients of $\cos x$:

Your solution

Answer

$7A - 6B = 3$

Also, equate coefficients of $\sin x$:

Your solution

Answer

$7B + 6A = 0$

Solve these two equations in A and B simultaneously to find values for A and B, and hence obtain the particular integral:

Your solution

Answer

$A = \frac{21}{85},\ B = -\frac{18}{85},\ y_{\mathsf{p}}(x) = \frac{21}{85}\cos x - \frac{18}{85}\sin x$

5. Finding the general solution of a second order linear inhomogeneous ODE

The general solution of a second order linear inhomogeneous equation is the sum of its particular integral and the complementary function. In subsection 2 (page 32) you learned how to find a complementary function, and in subsection 4 (page 42) you learnt how to find a particular integral. We now put these together to find the general solution.

Example 15

Find the general solution of $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 10y = 3x^2$

Solution

The complementary function was found in Example 8 to be $y_{\mathsf{cf}} = A\mathrm{e}^{2x} + B\mathrm{e}^{-5x}$.

The particular integral is found by trying a solution of the form $y = ax^2 + bx + c$, so that

$$\frac{dy}{dx} = 2ax + b, \qquad \frac{d^2y}{dx^2} = 2a$$

Substituting into the differential equation gives

$$2a + 3(2ax + b) - 10(ax^2 + bx + c) = 3x^2$$

Comparing constants: $\quad 2a + 3b - 10c = 0$

Comparing x terms: $\quad 6a - 10b = 0$

Comparing x^2 terms: $\quad -10a = 3$

So $\quad a = -\dfrac{3}{10}, \; b = -\dfrac{9}{50}, \; c = -\dfrac{57}{500}, \qquad y_{\mathsf{p}}(x) = -\dfrac{3}{10}x^2 - \dfrac{9}{50}x - \dfrac{57}{500}.$

Thus the general solution is $\quad y = y_{\mathsf{p}}(x) + y_{\mathsf{cf}}(x) = -\dfrac{3}{10}x^2 - \dfrac{9}{50}x - \dfrac{57}{500} + A\mathrm{e}^{2x} + B\mathrm{e}^{-5x}$

Key Point 9

The general solution of a second order constant coefficient ordinary differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \qquad \text{is} \qquad y = y_{\mathsf{p}} + y_{\mathsf{cf}}$$

being the sum of the particular integral and the complementary function.

y_{p} contains no arbitrary constants; y_{cf} contains two arbitrary constants.

Engineering Example 2

An LC circuit with sinusoidal input

The differential equation governing the flow of current in a series LC circuit when subject to an applied voltage $v(t) = V_0 \sin \omega t$ is $L\dfrac{d^2i}{dt^2} + \dfrac{1}{C}i = \omega V_0 \cos \omega t$

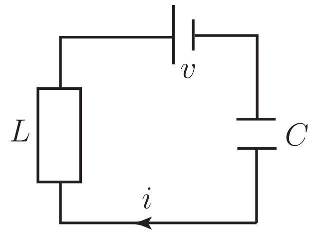


Figure 3

Obtain its general solution.

Solution

The homogeneous equation is $L\dfrac{d^2 i_{\mathsf{cf}}}{dt^2} + \dfrac{i_{\mathsf{cf}}}{C} = 0.$

Letting $i_{\mathsf{cf}} = \mathrm{e}^{kt}$ we find the auxiliary equation is $Lk^2 + \frac{1}{C} = 0$ so that $k = \pm \mathrm{i}/\sqrt{LC}$. Therefore, the complementary function is:

$$i_{\mathsf{cf}} = A\cos\frac{t}{\sqrt{LC}} + B\sin\frac{t}{\sqrt{LC}} \qquad \text{where } A \text{ and } B \text{ arbitrary constants.}$$

To find a particular integral try $i_{\mathsf{p}} = E\cos\omega t + F\sin\omega t$, where E, F are constants. We find:

$$\frac{di_{\mathsf{p}}}{dt} = -\omega E\sin\omega t + \omega F\cos\omega t \qquad \frac{d^2 i_{\mathsf{p}}}{dt^2} = -\omega^2 E\cos\omega t - \omega^2 F\sin\omega t$$

Substitution into the inhomogeneous equation yields:

$$L(-\omega^2 E\cos\omega t - \omega^2 F\sin\omega t) + \frac{1}{C}(E\cos\omega t + F\sin\omega t) = \omega V_0 \cos\omega t$$

Equating coefficients of $\sin\omega t$ gives: $-\omega^2 LF + (F/C) = 0.$

Equating coefficients of $\cos\omega t$ gives: $-\omega^2 LE + (E/C) = \omega V_0.$

Therefore $F = 0$ and $E = CV_0\omega/(1-\omega^2 LC)$. Hence the particular integral is

$$i_{\mathsf{p}} = \frac{CV_0\omega}{1-\omega^2 LC}\cos\omega t.$$

Finally, the general solution is:

$$i = i_{\mathsf{cf}} + i_{\mathsf{p}} = A\cos\frac{t}{\sqrt{LC}} + B\sin\frac{t}{\sqrt{LC}} + \frac{CV_0\omega}{1-\omega^2 LC}\cos\omega t$$

6. Inhomogeneous term in the complementary function

Occasionally you will come across a differential equation $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=f(x)$ for which the inhomogeneous term, $f(x)$, forms part of the complementary function. One such example is the equation

$$\frac{d^2y}{dx^2}-\frac{dy}{dx}-6y=\mathrm{e}^{3x}$$

It is straightforward to check that the complementary function is $y_{\mathsf{cf}}=A\mathrm{e}^{3x}+B\mathrm{e}^{-2x}$. Note that the first of these terms has the same form as the inhomogeneous term, e^{3x}, on the right-hand side of the differential equation.

You should verify for yourself that trying a particular integral of the form $y_{\mathsf{p}}(x)=\alpha\mathrm{e}^{3x}$ will not work in a case like this. Can you see why?

Instead, try a particular integral of the form $y_{\mathsf{p}}(x)=\alpha x\mathrm{e}^{3x}$. Verify that

$$\frac{dy_{\mathsf{p}}}{dx}=\alpha\mathrm{e}^{3x}(3x+1) \qquad \text{and} \qquad \frac{d^2y_{\mathsf{p}}}{dx^2}=\alpha\mathrm{e}^{3x}(9x+6).$$

Substitute these expressions into the differential equation to find $\alpha=\frac{1}{5}$.

Finally, the particular integral is $y_{\mathsf{p}}(x)=\frac{1}{5}x\mathrm{e}^{3x}$ and so the general solution to the differential equation is:

$$y=A\mathrm{e}^{3x}+B\mathrm{e}^{-2x}+\tfrac{1}{5}x\mathrm{e}^{3x}$$

This shows a generally effective method - where the inhomogeneous term $f(x)$ appears in the complementary function use as a trial particular integral x times what would otherwise be used.

Key Point 10

When solving

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

if the inhomogeneous term $f(x)$ appears in the complementary function use as a trial particular integral x times what would otherwise be used.

Exercises

1. Find the general solution of the following equations:

(a) $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} - 3x = 6$ (b) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 4y = 8$ (c) $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 6y = 2t$

(d) $\dfrac{d^2x}{dt^2} + 11\dfrac{dx}{dt} + 30x = 8t$ (e) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 3y = 2\sin 2x$ (f) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 4\cos 3t$

(g) $\dfrac{d^2y}{dx^2} + 9y = 4\mathrm{e}^{8x}$ (h) $\dfrac{d^2x}{dt^2} - 16x = 9\mathrm{e}^{6t}$

2. Find a particular integral for the equation $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 5\mathrm{e}^{3t}$

3. Find a particular integral for the equation $\dfrac{d^2x}{dt^2} - x = 4\mathrm{e}^{-2t}$

4. Obtain the general solution of $y'' - y' - 2y = 6$

5. Obtain the general solution of the equation $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 10\cos 2x$

   Find the particular solution satisfying $y(0) = 1,\ \dfrac{dy}{dx}(0) = 0$

6. Find a particular integral for the equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 1 + x$

7. Find the general solution of

   (a) $\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + 5x = 3$ (b) $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + x = \mathrm{e}^t$

Answers

1. (a) $x = A\mathrm{e}^{-t} + B\mathrm{e}^{3t} - 2$ (b) $y = A\mathrm{e}^{-x} + B\mathrm{e}^{-4x} + 2$ (c) $y = A\mathrm{e}^{-2t} + B\mathrm{e}^{-3t} + \frac{1}{3}t - \frac{5}{18}$

   (d) $x = A\mathrm{e}^{-6t} + B\mathrm{e}^{-5t} + 0.267t - 0.0978$

   (e) $y = \mathrm{e}^{-x}[A\sin\sqrt{2}x + B\cos\sqrt{2}x] - \frac{8}{17}\cos 2x - \frac{2}{17}\sin 2x$

   (f) $y = \mathrm{e}^{-0.5t}(A\cos 0.866t + B\sin 0.866t) - 0.438\cos 3t + 0.164\sin 3t$

   (g) $y = A\cos 3x + B\sin 3x + 0.0548\mathrm{e}^{8x}$ (h) $x = A\mathrm{e}^{4t} + B\mathrm{e}^{-4t} + \frac{9}{20}\mathrm{e}^{6t}$

2. $x_\mathsf{p} = 2.5\mathrm{e}^{3t}$

3. $x_\mathsf{p} = \frac{4}{3}\mathrm{e}^{-2t}$

4. $y = A\mathrm{e}^{2x} + B\mathrm{e}^{-x} - 3$

5. $y = A\mathrm{e}^{-2x} + B\mathrm{e}^{-x} + \frac{3}{2}\sin 2x - \frac{1}{2}\cos 2x;$ $y = \frac{3}{2}\mathrm{e}^{-2x} + \frac{3}{2}\sin 2x - \frac{1}{2}\cos 2x$

6. $y_\mathsf{p} = x$

7. (a) $x = A\mathrm{e}^{t} + B\mathrm{e}^{5t} + \frac{3}{5}$ (b) $x = A\mathrm{e}^{t} + Bt\mathrm{e}^{t} + \frac{1}{2}t^2\mathrm{e}^{t}$

Applications of Differential Equations 19.4

Introduction

Sections 19.2 and 19.3 have introduced several techniques for solving commonly occurring first-order and second-order ordinary differential equations. In this Section we solve a number of these equations which model engineering systems.

Prerequisites

Before starting this Section you should ...

- understand what is meant by a differential equation
- be familiar with the terminology associated with differential equations: order, dependent variable and independent variable
- be able to integrate standard functions

Learning Outcomes

On completion you should be able to ...

- recognise and solve first-order ordinary differential equations, modelling simple electrical circuits, projectile motion and Newton's law of cooling
- recognise and solve second-order ordinary differential equations with constant coefficients modelling free electrical and mechanical oscillations
- recognise and solve second-order ordinary differential equations with constant coefficients modelling forced electrical and mechanical oscillations

1. Modelling with first-order equations

Applying Newton's law of cooling

In Section 19.1 we introduced Newton's law of cooling. The model equation is

$$\frac{d\theta}{dt} = -k(\theta - \theta_{\mathsf{s}}) \qquad \theta = \theta_0 \ \text{ at } \ t = 0. \tag{5}$$

where $\theta = \theta(t)$ is the temperature of the cooling object at time t, θ_{s} the temperature of the environment (assumed constant) and k is a thermal constant related to the object, θ_0 is the initial temperature of the liquid.

Task

Solve this initial value problem:

$$\frac{d\theta}{dt} = -k(\theta - \theta_s), \qquad \theta = \theta_0 \quad \text{at} \quad t = 0$$

Separate the variables to obtain an equation connecting two integrals:

Your solution

Answer

$$\int \frac{d\theta}{\theta - \theta_{\mathsf{s}}} = -\int k\ dt$$

Now integrate both sides of this equation:

Your solution

Answer

$\ln(\theta - \theta_{\mathsf{s}}) = -kt + C$ where C is constant

Apply the initial condition and take exponentials to obtain a formula for θ:

Your solution

Answer

$\ln(\theta_0 - \theta_{\mathsf{s}}) = C$. Hence $\ln(\theta - \theta_{\mathsf{s}}) = -kt + \ln(\theta_0 - \theta_{\mathsf{s}})$ so that $\ln(\theta - \theta_{\mathsf{s}}) - \ln(\theta_0 - \theta_0) = -kt$

Thus, rearranging and inverting, we find:

$$\ln\left(\frac{\theta - \theta_{\mathsf{s}}}{\theta_0 - \theta_{\mathsf{s}}}\right) = -kt \qquad \therefore \qquad \frac{\theta - \theta_{\mathsf{s}}}{\theta_0 - \theta_{\mathsf{s}}} = \mathsf{e}^{-kt} \quad \text{giving} \quad \theta = \theta_{\mathsf{s}} + (\theta_0 - \theta_{\mathsf{s}})\mathsf{e}^{-kt}.$$

The graph of θ against t for $\theta = \theta_s + (\theta_0 - \theta_s)\mathrm{e}^{-kt}$ is shown in Figure 4 below.

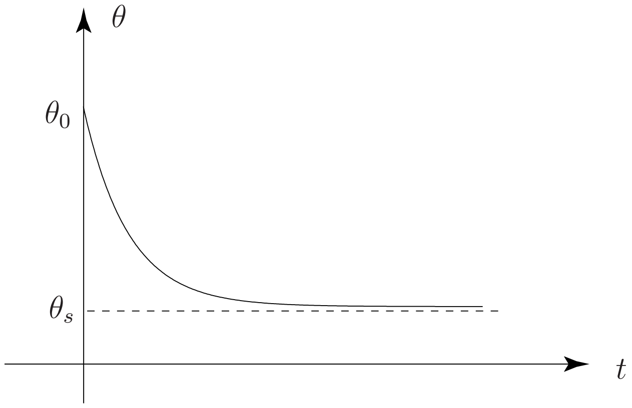


Figure 4

We see that as time increases ($t \to \infty$), then the temperature of the object cools down to that of the environment, that is: $\theta \to \theta_{\mathsf{s}}$.

We could have solved (5) by the integrating factor method, which you are now asked to do.

Task

We can write the equation for Newton's law of cooling (5) as

$$\frac{d\theta}{dt} + k\,\theta = k\theta_{\mathsf{s}}, \qquad \theta = \theta_0 \text{ at } t = 0 \tag{6}$$

State the integrating factor for this equation:

Your solution

Answer

$\mathrm{e}^{\int k\,dt} = \mathrm{e}^{kt}$ is the integrating factor.

Multiplying (6) by this factor we find that

$$\mathrm{e}^{kt}\frac{d\theta}{dt} + k\mathrm{e}^{kt}\theta = k\theta_{\mathsf{s}}\mathrm{e}^{kt} \qquad \text{or, rearranging,} \qquad \frac{d}{dt}(\mathrm{e}^{kt}\theta) = k\theta_{\mathsf{s}}\mathrm{e}^{kt}$$

Now integrate this equation and apply the initial condition:

Your solution

Answer

Integration produces $\mathrm{e}^{kt}\theta = \theta_{\mathsf{s}}\mathrm{e}^{kt} + C$, where C is an arbitrary constant. Then, applying the initial condition: when $t = 0$, $\theta_0 = \theta_{\mathsf{s}} + C$ so that $C = \theta_0 - \theta_{\mathsf{s}}$ gives the same result as before:

$$\theta = \theta_{\mathsf{s}} + (\theta_0 - \theta_{\mathsf{s}})\mathrm{e}^{-kt},$$

Modelling electrical circuits

Another application of first-order differential equations arises in the modelling of electrical circuits. In Section 19.1 the differential equation for the RL circuit in Figure 5 below was shown to be

$$L\,\frac{di}{dt} + Ri = E$$

in which the initial condition is $i = 0$ at $t = 0$.

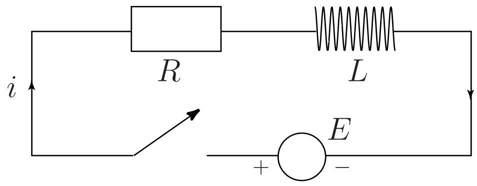


Figure 5

First we write this equation in standard form $\{\dfrac{dy}{dx} + P(x)y = Q(x)\}$ and obtain the integrating factor.

Dividing the differential equation through by L gives

$$\frac{di}{dt} + \frac{R}{L}\,i = \frac{E}{L}$$

which is now in standard form. The integrating factor is $\mathrm{e}^{\int \frac{R}{L}dt} = \mathrm{e}^{Rt/L}$.
Multiplying the equation in standard form by the integrating factor gives

$$\mathrm{e}^{Rt/L}\frac{di}{dt} + \mathrm{e}^{Rt/L}\frac{R}{L}i = \frac{E}{L}\mathrm{e}^{Rt/L}$$

or, rearranging,

$$\frac{d}{dt}(\mathrm{e}^{Rt/L}\ i) = \frac{E}{L}\mathrm{e}^{Rt/L}.$$

Now we integrate both sides and apply the initial condition to obtain the solution.

Integrating the differential equation gives:

$$\mathrm{e}^{Rt/L}\ i = \frac{E}{R}\ \mathrm{e}^{Rt/L} + C$$

where C is a constant so

$$i = \frac{E}{R} + C\mathrm{e}^{-Rt/L}$$

Applying the initial condition $i = 0$ when $t = 0$ gives

$$0 = \frac{E}{R} + C$$

so that $C = -\dfrac{E}{R}$.

Finally, $i = \dfrac{E}{R}(1 - \mathrm{e}^{-Rt/L})$.

Note that as $t \to \infty$, $i \to \dfrac{E}{R}$ so as t increases the effect of the inductor diminishes to zero.

Task

A spherical pill with volume V and surface area S is swallowed and slowly dissolves in the stomach, releasing an active component. In one model it is assumed that the capsule dissolves in the stomach acids such that the rate of change in volume, $\dfrac{dV}{dt}$, is directly proportional to the pill's surface area.

(a) Show that $\dfrac{dV}{dt} = -kV^{2/3}$ where k is a positive real constant and solve this given that $V = V_0$ at $t = 0$.

(b) Experimental measurements indicate that for a 4 mm pill, half of the volume has dissolved after 3 hours. Find the rate constant k (m s^{-1}).

(c) Estimate the time required for 95% of the pill to dissolve.

(a) First write down the formulae for volume of a sphere (V) and surface area of a sphere (S) and so express S in terms of V by eliminating r:

Your solution

Answer

$$V = \frac{4}{3}\pi r^3 \qquad\qquad S = 4\pi r^2$$

From the V equation $r = \left(\dfrac{3V}{4\pi}\right)^{1/3}$ so $S = (36\pi)^{1/3}V^{2/3} = kV^{2/3}$ for constant k.

Now write down the differential equation modelling the solution:

Your solution

Answer

$$\frac{dV}{dt} = -kV^{2/3} \qquad \text{(negative to represent a decrease with time)}$$

Using the condition $V = V_0$ when $t = 0$, solve the differential equation:

Your solution

Answer
Solving by separation of variables gives

$$V = \left\{\frac{1}{3}(C - kt)\right\}^{1/3}$$

and setting $V = V_0$ when $t = 0$ means

$V_0 = \left(\frac{1}{3}C\right)^3$ so $C = 3V_0^{1/3}$ and the solution is

$$V = \left\{V_0^{1/3} - \frac{kt}{3}\right\}^3$$

(b) Impose the condition that half the volume has dissolved after 3 hours to find k:

Your solution

Answer

$$V = \left\{V_0^{1/3} - \frac{kt}{3}\right\}^3$$

and when $t = 3$, $V = \frac{V_0}{2}$ so

$$\left(\frac{V_0}{2}\right)^{1/3} = V_0^{1/3} - k \qquad \text{and so} \qquad k = V_0^{1/3}(1 - (0.5)^{1/3})$$

(c) First write down the solution to the differential equation inserting the value of k obtained in (b) and then use it to estimate the time to 95% dissolving:

Your solution

Answer

$$V = \left\{ V_0^{1/3} - V_0^{1/3}(1-(0.5)^{1/3})\frac{t}{3} \right\}^3 \qquad \text{i.e.} \qquad V = V_0 \left\{ 1 - (1-(0.5)^{1/3})\frac{t}{3} \right\}^3$$

When 95% dissolved $V = 0.05V_0$ so

$$0.05V_0 = V_0 \left\{ 1 - (1-(0.5)^{1/3})\frac{t}{3} \right\}^3 \qquad \text{so} \qquad (0.05)^{1/3} = 1 - (1-(0.5)^{1/3})\frac{t}{3}$$

so

$$t = 3\left\{ \frac{1-(0.05)^{1/3}}{1-(0.5)^{1/3}} \right\} \approx 9.185 \approx 9 \text{ hr } 11 \text{ min}$$

2. Modelling free mechanical oscillations

Consider the following schematic diagram of a shock absorber:

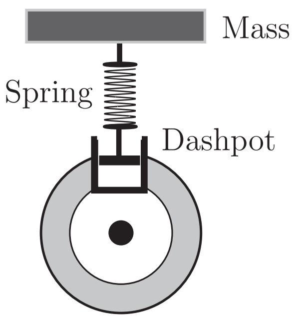


Figure 6

The equation of motion can be described in terms of the vertical displacement x of the mass.

Let m be the mass, $k\dfrac{dx}{dt}$ the damping force resulting from the dashpot and nx the restoring force resulting from the spring. Here, k and n are constants.

Then the equation of motion is

$$m\frac{d^2x}{dt^2} = -k\frac{dx}{dt} - nx.$$

Suppose that the mass is displaced a distance x_0 initially and released from rest. Then at $t = 0$, $x = x_0$ and $\dfrac{dx}{dt} = 0$. Writing the differential equation in standard form gives

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + nx = 0.$$

We shall see that the nature of the oscillations described by this differential equation depends crucially upon the relative values of the mechanical constants m, k and n. This will be explored in subsequent Tasks.

Task

Find and solve the auxiliary equation of the differential equation

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + nx = 0.$$

Your solution

Answer

Putting $x = \mathrm{e}^{\lambda t}$, the auxiliary equation is $\quad m\,\lambda^2 + k\,\lambda + n = 0.$

Hence $\lambda = \dfrac{-k \pm \sqrt{k^2 - 4m\,n}}{2m}$.

The value of k controls the amount of damping in the system. We explore the solution for various values of k.

Case 1: No damping

If $k = 0$ then there is no damping. We expect, in this case, that once motion has started it will continue for ever. The motion that ensues is called **simple harmonic motion**. In this case we have

$$\lambda = \frac{\pm\sqrt{-4m\,n}}{2m}, \quad \text{that is,} \quad \lambda = \pm\sqrt{\frac{n}{m}}\,\mathrm{i} \qquad \text{where} \quad \mathrm{i}^2 = -1.$$

and the solution for the displacement x is:

$$x = A\cos\left(\sqrt{\frac{n}{m}}\,t\right) + B\sin\left(\sqrt{\frac{n}{m}}\,t\right) \quad \text{where } A, B \text{ are arbitrary constants.}$$

Task

Impose the initial conditions $x = x_0$ and $\dfrac{dx}{dt} = 0$ at $t = 0$ to find the unique solution to the ODE:

Your solution

Answer

$$\frac{dx}{dt} = -\sqrt{\frac{n}{m}}\ A\sin\left(\sqrt{\frac{n}{m}}\ t\right) + \sqrt{\frac{n}{m}}B\cos\left(\sqrt{\frac{n}{m}}\ t\right)$$

When $t = 0,\ \dfrac{dx}{dt} = 0$ so that $\quad \sqrt{\dfrac{n}{m}}\ B = 0 \quad$ so that $\quad B = 0.$

Therefore $\quad x = A\cos\left(\sqrt{\dfrac{n}{m}}\ t\right).$

Imposing the remaining initial condition: when $t = 0,\ x = x_0$ so that $x_0 = A$ and finally:

$$x = x_0\cos\left(\sqrt{\frac{n}{m}}\ t\right).$$

Case 2: Light damping

If $k^2 - 4mn < 0$, i.e. $k^2 < 4mn$ then the roots of the auxiliary equation are complex:

$$\lambda_1 = \frac{-k + \mathrm{i}\sqrt{4mn - k^2}}{2m} \qquad \lambda_2 = \frac{-k - \mathrm{i}\sqrt{4mn - k^2}}{2m}$$

Then, after some rearrangement:

$$x = \mathrm{e}^{-kt/2m}\left[A\cos pt + B\sin pt\right] \quad \text{in which} \quad p = \sqrt{4mn - k^2}/2m.$$

Task

If $m = 1,\ n = 1$ and $k = 1$ find λ_1 and λ_2 and then find the solution for the displacement x.

Your solution

Answer

$\lambda = \dfrac{-1 + \mathrm{i}\sqrt{4-1}}{2} = -1/2 \pm \mathrm{i}\sqrt{3}/2$. Hence $x = \mathrm{e}^{-t/2}\left[A\cos\dfrac{\sqrt{3}}{2}t + B\sin\dfrac{\sqrt{3}}{2}t\right]$.

Impose the initial conditions $x = x_0,\ \dfrac{dx}{dt} = 0$ at $t = 0$ to find the arbitrary constants and hence find the solution to the ODE:

Your solution

Answer

Differentiating, we obtain

$$\frac{dx}{dt} = -\frac{1}{2}\mathrm{e}^{-t/2}\left[A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right] + \mathrm{e}^{-t/2}\left[-\frac{\sqrt{3}}{2}A\sin\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}B\cos\frac{\sqrt{3}}{2}t\right]$$

At $t = 0$,

$$x = x_0 = A \qquad \text{(i)}$$

$$\frac{dx}{dt} = 0 = -\frac{1}{2}A + \frac{\sqrt{3}}{2}B \qquad \text{(ii)}$$

Solving (i) and (ii) we obtain

$$A = x_0 \qquad B = \frac{\sqrt{3}}{3}x_0 \quad \text{then} \quad x = x_0\mathrm{e}^{-t/2}\left[\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t\right].$$

The graph of x against t is shown in Figure 7. This is the case of light damping. As the damping in the system decreases (i.e. $k \to 0$) the number of oscillations (in a given time interval) will increase. In many mechanical systems these oscillations are usually unwanted and the designer would choose a value of k to either reduce them or to eliminate them altogether. For the choice $k^2 = 4mn$, known as the **critical damping** case, all the oscillations are absent.

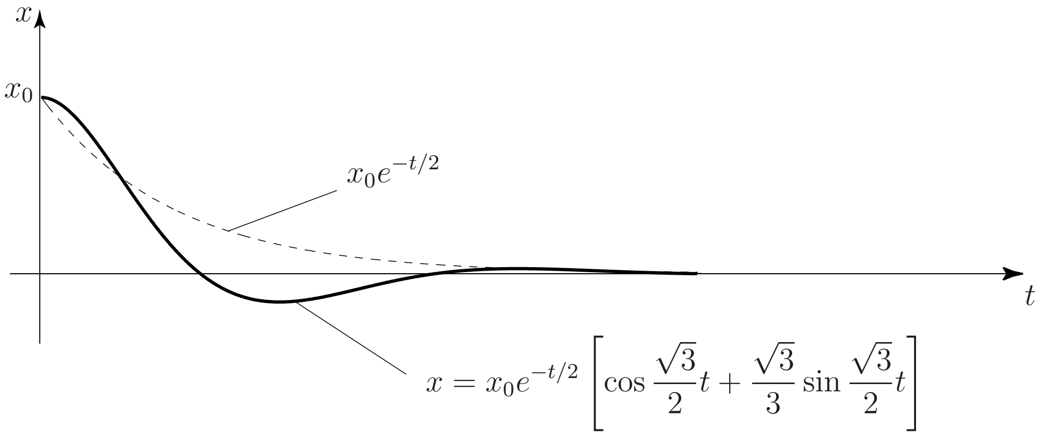


Figure 7

Case 3: Heavy damping

If $k^2 - 4mn > 0$, i.e. $k^2 > 4mn$, then there are two real roots of the auxiliary equation, λ_1 and λ_2:

$$\lambda_1 = \frac{-k + \sqrt{k^2 - 4mn}}{2m} \qquad \lambda_2 = \frac{-k - \sqrt{k^2 - 4mn}}{2m}$$

Then

$$x = A\mathrm{e}^{\lambda_1 t} + B\mathrm{e}^{\lambda_2 t}.$$

Task

If $m = 1,\ n = 1$ and $k = 2.5$ find λ_1 and λ_2 and then find the solution for the displacement x.

Your solution

Answer

$$\lambda = \frac{-2.5 \pm \sqrt{6.25 - 4}}{2} = -1.25 \pm 0.75$$

Hence $\lambda_1,\ \lambda_2 = -0.5, -2$ and so $x = A\mathrm{e}^{-0.5t} + B\mathrm{e}^{-2t}$

Impose the initial conditions $x = x_0,\ \dfrac{dx}{dt} = 0$ at $t = 0$ to find the arbitrary constants and hence find the solution to the ODE.

Your solution

Answer

Differentiating, we obtain

$$\frac{dx}{dt} = -0.5A\text{e}^{-0.5t} - 2B\text{e}^{-2t}$$

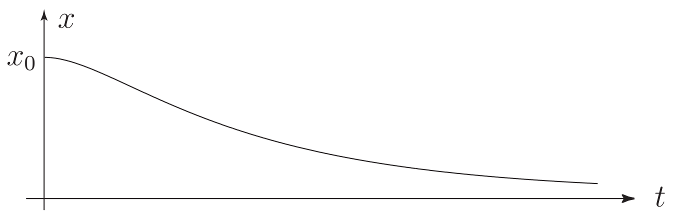
At $t = 0$,

$$x = x_0 = A + B \qquad \text{(i)}$$

$$\frac{dx}{dt} = 0 = -0.5A - 2B \qquad \text{(ii)}$$

Solving (i) and (ii) we obtain $\quad A = \dfrac{4}{3}x_0 \qquad B = -\dfrac{1}{3}x_0$ then $\quad x = \dfrac{1}{3}x_0(4\text{e}^{-0.5t} - \text{e}^{-2t})$.

The graph of x against t is shown below. This is the case of heavy damping.



Other cases are dealt with in the Exercises at the end of the Section.

3. Modelling forced mechanical oscillations

Suppose now that the mass is subject to a force $f(t)$ after the initial disturbance. Then the equation of motion is

$$m\frac{d^2x}{dt^2} + k\,\frac{dx}{dt} + nx = f(t)$$

Consider the case $f(t) = F\cos\omega t$, that is, an oscillatory force of magnitude F and angular frequency ω. Choosing specific values for the constants in the model: $m = n = 1$, $k = 0$, and $\omega = 2$ we find

$$\frac{d^2x}{dt^2} + x = F\cos 2t$$

Task

Find the complementary function for the differential equation

$$\frac{d^2x}{dt^2} + x = F\cos 2t$$

Your solution

Answer

The homogeneous equation is

$$\frac{d^2x}{dt^2} + x = 0$$

with auxiliary equation $\lambda^2 + 1 = 0$. Hence the complementary function is

$$x_{\mathsf{cf}} = A\cos t + B\sin t.$$

Now find a particular integral for the differential equation:

Your solution

Answer

Try $x_{\mathsf{p}} = C\cos 2t + D\sin 2t$ so that $\dfrac{d^2x_{\mathsf{p}}}{dt^2} = -4C\cos 2t - 4D\sin 2t$. Substituting into the differential equation gives

$$(-4C + C)\cos 2t + (-4D + D\ \sin 2t) \equiv F\cos 2t.$$

Comparing coefficients gives $-3C = F$ and $-3D = 0$ so that $D = 0,\quad C = -\dfrac{1}{3}F$ and $x_{\mathsf{p}} = -\dfrac{1}{3}F\cos 2t$. The general solution of the differential equation is therefore

$$x = x_{\mathsf{p}} + x_{\mathsf{cf}} = -\frac{1}{3}F\cos 2t + A\cos t + B\sin t.$$

Finally, apply the initial conditions to find the solution for the displacement x:

Your solution

Answer

We need to determine the derivative and apply the initial conditions:

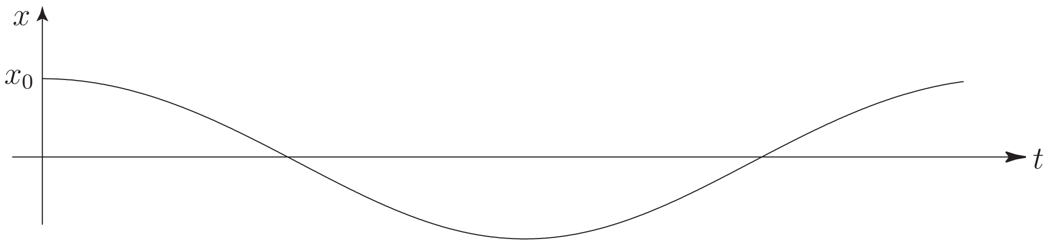
$$\frac{dx}{dt} = \frac{2}{3}F\sin 2t - A\sin t + B\cos t.$$

At $t = 0$ $\qquad x = x_0 = -\dfrac{1}{3}F + A$ and $\dfrac{dx}{dt} = 0 = B$

Hence $\qquad B = 0$ and $A = x_0 + \dfrac{1}{3}F.$

Then $\qquad x = -\dfrac{1}{3}F\cos 2t + \left(x_0 + \dfrac{1}{3}F\right)\cos t.$

The graph of x against t is shown below.



If the angular frequency ω of the applied force is *nearly equal to* that of the free oscillation the phenomenon of **beats** occurs. If the angular frequencies are *equal* we get the phenomenon of **resonance**. Note that we can eliminate resonance by introducing damping into the system.

4. Modelling forces on beams

Engineering Example 3

Shear force and bending moment of a beam

Introduction

The beam is a fundamental part of most structures we see around us. It may be used in many ways depending as how its ends are fixed. One end may be rigidly fixed and the other free (called **cantilevered**) or both ends may be resting on supports (called **simply supported**). Other combinations are possible. There are three basic quantities of interest in the deformation of beams, the deflection, the shear force and the bending moment.

For a beam which is supporting a load of w (measured in N m^{-1} and which may represent the self-weight of the beam or may be an external load), the shear force is denoted by S and measured in N m^{-1} and the bending moment is denoted by M and measured in N m^{-1}.

The quantities M, S and w are related by

$$\frac{dM}{dz} = S \qquad (1)$$

and

$$\frac{dS}{dz} = -w \qquad (2)$$

where z measures the position along the beam. If one of the quantities is known, the others can be calculated from the Equations (1) and (2). In words, the shear force is the negative of the derivative (with respect to position) of the bending moment and the load is the derivative of the shear force. Alternatively, the shear force is the negative of the integral (with respect to position) of the load and the bending moment is the integral of the shear force. The negative sign in Equation (2) reflects the fact that the load is normally measured positively in the downward direction while a positive shear force refers to an upward force.

Problem posed in words

A beam is fixed rigidly at one end and free to move at the other end (like a diving board). It only has to support its own weight. Find the shear force and the bending moment along its length.

Mathematical statement of problem

A uniform beam of length L, supports its own weight w_o (a constant). At one end $(z = 0)$, the beam is fixed rigidly while the other end $(z = L)$ is free to move. Find the shear force S and the bending moment M as functions of z.

Mathematical analysis

As w is a constant, Equation (2) gives

$$S = -\int w dz = -\int w_o dz = -w_o z + C.$$

At the free end $(z = L)$, the shear force $S = 0$ so $C = w_o L$ giving

$$S = w_o (L - z)$$

This expression can be substituted into Equation (1) to give

$$M = \int S dz = \int w_o (L - z)\, dz = \int (w_o L - w_o z)\, dz = w_o L z - \frac{w_o}{2} z^2 + K.$$

Once again, $M = 0$ at the free end $z = L$ so K is given by $K = -\dfrac{w_o}{2} L^2$. Thus

$$M = w_o L z - \frac{w_o}{2} z^2 - \frac{w_o}{2} L^2$$

The diagrams in Figure 8 show the load w (Figure 8a), the shear force S (Figure 8b) and the bending moment M (Figure 8c) as functions of position z.

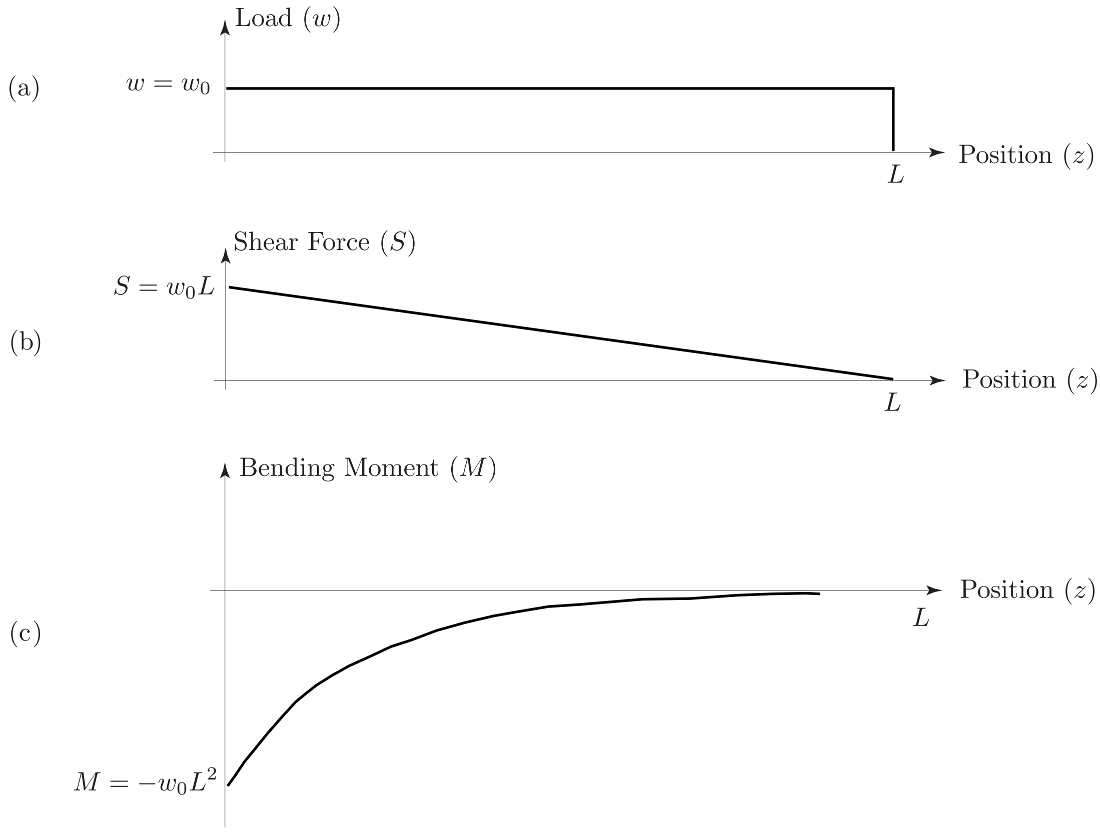


Figure 8: The loading (a), shear force (b) and bending moment (c) as functions of position z

Interpretation

The beam deforms (as we might have expected) with the shear force and bending moments having maximum values at the fixed end and minimum (zero) values at the free end. You can easily experience this for yourselves: simply hold a wooden plank (not too heavy) at one end with both hands so that it is horizontal. As you try this with planks of increasing length (and hence weight) you will find it increasingly difficult to support the weight of the plank (this is the shear force) and increasingly difficult to keep the plank horizontal (this is the bending moment).

This mathematical model is an excellent description of real beams.

Engineering Example 4

Deflection of a uniformly loaded beam

Introduction

A uniformly loaded beam of length L is supported at both ends as shown in Figure 9. The deflection $y(x)$ is a function of horizontal position x and obeys the ordinary differential equation (ODE)

$$\frac{d^4y}{dx^4}(x) = \frac{1}{EI}q(x) \qquad (1)$$

where E is Young's modulus, I is the moment of inertia and $q(x)$ is the load per unit length at point x. We assume in this problem that $q(x) = q$ a constant. The boundary conditions are (i) no deflection at $x = 0$ and $x = L$ (ii) no curvature of the beam at $x = 0$ and $x = L$.

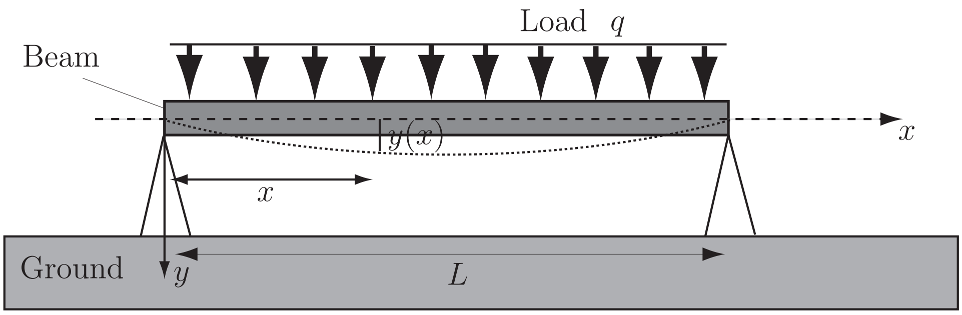


Figure 9: The bending beam, parameters involved in the mathematical formulation

Problem in words

Find the deflection of a beam, supported so that that there is no deflection and no curvature of the beam at its ends, subject to a uniformly distributed load, as a function of position along the beam.

Mathematical statement of problem

Find the equation of the curve $y(x)$ assumed by the bending beam that satisfies the ODE (1). Use the coordinate system shown in Figure 9 where the origin is at the left extremity of the beam. In this coordinate system, the boundary conditions, which require that there is no deflection at $x = 0$ and $x = L$, and that there is no curvature of the beam at $x = 0$ and $x = L$, are

(a) $y(0) = 0$

(b) $y(L) = 0$

(c) $\frac{d^2y}{dx^2}\Big|_{x=0} = 0$

(d) $\frac{d^2y}{dx^2}\Big|_{x=L} = 0$

(e) $\frac{d^2y}{dx^2}\Big|_{x=L} = 0$

Note that $\frac{dy(x)}{dx}$ and $\frac{d^2y(x)}{dx^2}$ are respectively the slope and the radius of curvature of the curve at point (x, y).

Mathematical analysis

Integrating Equation (1) leads to:

$$EI\frac{d^3y}{dx^3}(x) = qx + A \qquad (2)$$

Integrating a second time:

$$EI\frac{d^2y}{dx^2}(x) = qx^2/2 + Ax + B \qquad (3)$$

Integrating a third time:

$$EI\frac{dy}{dx}(x) = qx^3/6 + Ax^2/2 + Bx + C \qquad (4)$$

Integrating a fourth time:

$$EIy(x) = qx^4/24 + Ax^3/6 + Bx^2/2 + Cx + D. \qquad (5)$$

The boundary conditions (a) and (b) enable determination of the constants of integration A, B, C, D. Indeed, the boundary condition (a), $y(0) = 0$, and Equation (5) give

$$EIy(0) = q \times (0)^4/24 + A \times (0)^3/6 + B \times (0)^2/2 + C \times (0) + D = 0$$

which yields $D = 0$.
The boundary condition (b), $y(L) = 0$, and Equation (5) give

$$EIy(L) = qL^4/24 + AL^3/6 + BL^2/2 + CL + D.$$

Using the newly found value for D one writes

$$qL^4/24 + AL^3/6 + BL^2/2 + CL = 0 \qquad (6)$$

The boundary condition (c) obtained from the definition of the radius of curvature, $\frac{d^2y}{dx^2}(0) = 0$, and Equation (3) give

$$I\frac{d^2y}{dx^2}(0) = q \times (0)^2/2 + A \times (0) + B$$

which yields $B = 0$. The boundary condition (d), $\frac{d^2y}{dx^2}(L) = 0$, and Equation (3) give

$$EI\frac{d^2y}{dx^2}(L) = qL^2/2 + AL = 0$$

which yields $A = -qL/2$. The expressions for A, B, D are introduced in Equation (6) to find the last unknown constant C. This leads to $qL^4/24 - qL^4/12 + CL = 0$ or $C = qL^3/24$. Finally, Equation (5) and the values of constants lead to the solution

$$y(x) = [qx^4/24 - qLx^3/12 + qL^3x/24]/EI. \qquad (7)$$

Interpretation

The predicted deflection is zero at both ends as required, and you may check that it is symmetrical about the centre of the beam by switching to the coordinate system (X, Y) with $L/2 - x = X$ and $y = Y$ and verifying that the deflection $Y(X)$ is symmetrical about the vertical axis, i.e. $Y(X) = Y(-X)$.

Exercises

1. In an RC circuit (a resistor and a capacitor in series) the applied emf is a constant E. Given that $\dfrac{dq}{dt} = i$ where q is the charge in the capacitor, i the current in the circuit, R the resistance and C the capacitance the equation for the circuit is

   $$Ri + \frac{q}{C} = E.$$

   If the initial charge is zero find the charge subsequently.

2. If the voltage in the RC circuit is $E = E_0 \cos \omega t$ find the charge and the current at time t.

3. An object is projected from the Earth's surface. What is the least velocity (the escape velocity) of projection in order to escape the gravitational field, ignoring air resistance.

   The equation of motion is

   $$m\,v\frac{dv}{dx} = -m\,g\frac{R^2}{x^2}$$

   where the mass of the object is m, its distance from the centre of the Earth is x and the radius of the Earth is R.

4. The radial stress p at distance r from the axis of a thick cylinder subjected to internal pressure is given by $p + r\dfrac{dp}{dr} = A - p$ where A is a constant. If $p = p_0$ at the inner wall $(r = r_1)$ and is negligible $(p = 0)$ at the outer wall $(r = r_2)$ find an expression for p.

5. The equation for an LCR circuit with applied voltage E is

   $$L\frac{di}{dt} + Ri + \frac{1}{C}q = E.$$

   By differentiating this equation find the solution for $q(t)$ and $i(t)$ if $L = 1$, $R = 100$, $C = 10^{-4}$ and $E = 1000$ given that $q = 0$ and $i = 0$ at $t = 0$.

6. Consider the free vibration problem in Section 19.4 subsection 2 (page 57) when $m = 1,\ n = 1$ and $k = 2$ (critical damping).

   Find the solution for $x(t)$.

7. Repeat Exercise 6 for the case $m = 1,\ n = 1$ and $k = 1.5$ (light damping)

8. Consider the forced vibration problem in Section 19.4 subsection 2 with $m = 1,\ n = 25,\ k = 8,\ E = \sin 3t,\ x_0 = 0$ with an initial velocity of 3.

9. This refers to the Task on page 55 concerning modelling the dissolving of a pill in the stomach.

   An alternative model supposes that the pill is very rapidly permeated by stomach acids and the small granules contained in the capsule dissolve individually. In this case, the rate of change of volume is assumed to be directly proportional to the volume. Using the experimental data given in the Task, estimate the time for 95% of the pill to dissolve, based on this alternative model, and compare results.

Answers

1. Use the equation in the form $R\dfrac{dq}{dt} + \dfrac{q}{C} = E$ or $\dfrac{dq}{dt} + \dfrac{1}{RC}q = \dfrac{E}{R}$.

   The integrating factor is $\mathrm{e}^{t/RC}$ and the general solution is

   $q = EC(1 - \mathrm{e}^{-t/RC})$ and as $t \to \infty \quad q \to EC$.

2. $q = \dfrac{E_0 C}{1 + \omega^2 R^2 C^2} \left[\cos\omega t - \mathrm{e}^{-t/RC} + \omega RC \sin\omega t\right]$

   $i = \dfrac{dq}{dt} = \dfrac{E_0 C}{1 + \omega^2 R^2 C^2} \left[-\omega \sin\omega t + \dfrac{1}{RC}\ \mathrm{e}^{-t/RC} + \omega^2 RC \cos\omega t\right].$

3. $v_{\mathsf{min}} = \sqrt{2gR}$. If $R = 6378$ km and $g = 9.81$ m s^{-2} then $v_{\mathsf{min}} = 11.2$ km s^{-1}.

4. $p = \dfrac{p_0 r_1^2}{r_1^2 - r_2^2}\left(1 - \dfrac{r_2^2}{r^2}\right)$

5. $q = 0.1 - \dfrac{1}{10\sqrt{3}}\mathrm{e}^{-50t}(\sin 50\sqrt{3}t + \sqrt{3}\cos 50\sqrt{3}t) \qquad i = \dfrac{20}{\sqrt{3}}\mathrm{e}^{-50t}\sin 50\sqrt{3}t.$

6. $x = x_0(1 + t)e^{-t}$

7. $x = x_0 e^{-0.75t}(\cos\dfrac{\sqrt{7}}{4}t + \dfrac{3}{\sqrt{7}}\sin\dfrac{\sqrt{7}}{4}t)$

8. $x = \dfrac{1}{104}\left[\mathrm{e}^{-4t}\left(3\cos 3t + 106\sin 3t\right) - 3\cos 3t + 2\sin 3t)\right]$

9. This leads to $\dfrac{dV}{dt} = -kV$ and $V = V_0\mathrm{e}^{-kt}$ where $k = \dfrac{1}{3}\ln 2$. The time taken is about 4 hr 19 min. This is much less than the other model, as should be expected.